

Resource Dependent Branching Processes and the Envelope of Societies

F. Thomas Bruss and Mitia Duerinckx
Université Libre de Bruxelles

Abstract

Why did mankind create, and continues to create, societies? What is it that seems to keep societies together? And then, are there natural boundaries for societies mankind would not exceed?

The first question is rather philosophical, and we comment only briefly on it. The second one displays several quantifiability aspects which would be open to statistical hypothesis testing, but we only treat it in as much as it concerns the third question, which attracts our particular interest. Here we shall provide a *mathematical* answer obtained from what we propose as a global mathematical model for societies. It is built on branching processes, and submitted to two natural hypotheses. Still rudimentary, our model allows nevertheless to take into account essential features of life within a society. Apart from reproduction of individuals, it incorporates the following factors and interdependencies: the desire to survive, heritage and production of resources, consumption of resources, policies to distribute resources among individuals, and, decisively, the right of emigration. We look at different submodels of the model, characterising different societies. These are defined by the type of control they exercise through different policies to distribute resources among their individuals.

Our main results are: Firstly, in the global model we consider, the answer to the third question is affirmative. Secondly, we can determine these boundaries. Thirdly, we can also determine their critical parameters, and interpret them. Indeed, there are exactly two societies which form an *envelope* in the sense that all societies have to live in the long run between these two boundaries. We call the boundaries *weakest-first* society and *strongest-first* society. Interestingly, each is related with a society form we believe to recognise rather well. It seems that, with respect to societies, mankind has already come close to testing the limits.

Keywords: Asexual Galton-Watson processes, controlled branching processes, extinction criteria, almost-sure convergence, stochastic order, order statistics, expected stopping times, Lorenz curve, society structure, Darwinism, Communism, Capitalism.

AMS subject classification: primary 69J85, 60J05; secondary 60G40

Running title: Branching Processes and Societies

1 Introduction

Different theories in anthropology and sociology suggest that man has a natural tendency for socialising. Some of these theories see this tendency as a fact which is, *a priori*, independent of the idea of synergy effects and utility. Others contrast in the sense that they consider the essential reasons why people would join together to form groups or larger communities lie in the desire to profit from an efficient collaboration, as e.g. in hunting, defense, agricultural activities, industrial production, etc. As so often, the truth may lie somewhere in the middle. But then for any theory explaining the phenomenon of socialising, other questions come up naturally: Why do groups and larger communities have a tendency to go further and develop “structured” societies with, often enough, very restrictive rules? And then, how far can man go? Are there natural boundaries for societies?

Certain aspects of these questions are equally interesting for animal societies. In fact, throughout this paper we shall always speak of “individuals” to make clear that, although the motivation stems from thinking about man, we keep general populations in mind.

The present paper tries to add to our understanding by looking at certain questions which naturally arise from basic mathematical models. We link a population model (in terms of a branching process model) with different rules to distribute resources without which the members of the population cannot live and reproduce. The philosophy behind the model we propose is that if mankind has a natural tendency for socialising, then it must also have the desire to see this social environment survive. With limited life time for each individual, survival means survival of their descendants. Moreover, to maintain a society, sufficiently many individuals in the population should have access to a minimum standard of living.

Theoretical desiderata must here be adapted to reality. So, for instance, in branching process models, survival is usually seen as survival “forever”. We know that with our current understanding of life in our solar system the “forever” is bound to have the meaning of “very long.” Also, in population models based on branching processes, a typical feature is that the population will either get extinct or, alternatively, explode, the latter being again incompatible with reality. Nevertheless, similarly as in modelling physical phenomena, where extrapolation often makes no sense outside certain thresholds but gives interesting insight within certain bounds, our models may be seen as adequate as a first approach.

The first-named author has been thinking about such problems repeatedly for many years. Much effort has gone into constructing a tractable model. He had spoken about *resource dependent branching processes* as early as 1983, given a second talk on this around 1995, and a third one around 2002. Seeing his results as being immature, he had never submitted any of them for publication. As far as we are aware, although in the meantime several interesting new branching process models have been created, this type of interdependent models has not been studied. Hence the models and results presented in this paper are believed to be new, and, now refined, of true interest.

2 Objective and content of this paper

Our objective is to answer the questions we asked through a suitable modeling and mathematical conclusions. Our approach is to construct branching process models in which individuals have to create resources in order to be able to live and where the control is implemented by policies to distribute resources among individuals.

The distribution of resources is effected in each generation, the planning is for the longer-time future development of the population, however. We suppose that individuals inherit resources from preceding generations, consume resources, and create new resources. The resources an individual can use during its lifetime are supposed to determine its standard of living. However, for simplicity, we model this relationship between individually available resources and the individual perception of standard of living only in a “binary” way, as will be explained below after the main hypotheses.

The inherited resources, plus the newly created ones, are considered as being the individual’s contribution to society. We model heritage and production (combined) of resources of individuals as independent random variables with a common continuous distribution function.

Our interest focusses on the implications of the following two basic hypotheses on the policy by which the population distributes its resources:

Hypothesis 1: Populations want to survive;

Hypothesis 2: Individuals prefer a higher standard of living to a lower one.

The desire to survive is here understood as the objective to have for the society as a whole a positive probability of surviving forever (in the strict sense). We suppose that populations determine rules for distributing resources to individuals who have random resource claims. If certain rules allow for a positive probability of survival whereas other rules would not achieve this, then the objective to

survive is taking priority, and the rules would be changed accordingly.

Apart from the reproduction mean per individual and the resource production mean per individual, the distribution of resource claims and the rules to try to satisfy these claims will play an equally important rule for survival. We think of each claim as being the outcome of what an individual, with its own power of conviction, will be able to defend within the society. The random claims of individuals are also modelled as independent random variables governed by a known continuous distribution function. The idea is of course that the latter may be chosen as a function of the distribution of resource production. For our objective in the present paper it will however suffice to consider both distributions as fixed.

The effect of the interplay of claims and available resources is modelled as follows. Each individual is supposed to emigrate if and only if its individual resource claim is not completely satisfied; otherwise it remains a member of the population until the end of the generation. Emigration is supposed to happen before an individual produces offspring. Hence the resource assignment (offered standard of living) is seen by an individual as being either sufficient, implying “stay”, or else insufficient, implying “leave”. This is the “binary” measure of satisfaction we announced above.

The reproduction of the population itself is assumed to be in accordance to the rules of a Galton-Watson branching process (GWP), i.e. members of the population reproduce independently of each other with the same distribution of the number of offspring.

Related work

Our model is an asexual controlled branching process (BP), where *controlled* should be understood in a larger sense. The general control is governed by a population-size dependent function of sums of dependent variables, and self-imposed. This strong dependence property excludes the powerful generating function machinery, of course. Moreover, although still rudimentary, the model seems no longer tractable for other strong tools as e.g. martingale arguments.

As we understand, and explain below, our model does not fit other models studied in the literature. Nevertheless, related work is sincerely acknowledged for having helped to get a feeling what result one can, or cannot, possibly hope for.

The model is neither a BP with varying environment (see e.g. Cohn [1996]) nor a BP with random environment. See Jagers [1975] for a clear analysis of the connection between these two types, and e.g. Haccou, Jagers, and Vatutin [2007] for newer developments. Our model is no multi-type BP model, and neither a pure population size-dependent model. It is a Markov process, as we shall see, but no phase-type Markov model or decomposable BP (see Hautphenne [2012]) can play the control we have in mind.

Early work on controlled branching processes confined interest to control through bounds imposed on the growth of GW-type processes. Sevast’janov and Zubkov [1974], Schuh [1976] and others modified the number of individuals which are allowed to reproduce in each generation by corresponding deterministic functions. Bruss [1978] considered a GWP with a non-specified absorbing process for which only the expected influence is known. Yanev [1976] studied so-called ϕ -branching processes where the growth of the GWP reproduction is controlled by random numbers of offspring which are allowed to reproduce. A similar model for random control functions was studied in Bruss [1980], and in more generality by González, Molina, and Del Puerto [2002]. The same authors also examined L_2 -convergence for such processes (see González, Molina, and del Puerto [2005]). Population-size dependence is another interesting access to control in branching process models. These were studied by Klebaner [1985] and Cohn and Klebaner [1986]. Xu and Mannor [2012] proposed a special class of controlled branching processes involving the notion of “resources”. Motivated by applications in marketing, their objective is to control independent subpopulations (multi-type model) in such a way that they grow as quickly as possible under an (optimal) assignment of resources. Relative frequencies of types were studied in Yakovlev and Yanev [2009].

3 Mathematical Model

We start by a formal definition of the global model which will define the common structure of all the models we consider. Here we postpone the explanations which will become clear as soon as we introduce those specific models which attract our particular interest.

Let $(D_n^k, X_n^k, R_n^k)_{n \in \mathbb{N}, k \in \mathbb{N}_0}$ be a double array of triplets of random variables defined on a probability space (Ω, \mathcal{F}, P) , where all triplets (D_n^k, X_n^k, R_n^k) are i.i.d. with the same distribution as some triplet of random variables (D, X, R) .

D_n^k , X_n^k and R_n^k ($k \in \mathbb{N}_0$) represent the number of offspring, the resource claims and the production of resources (respectively) of each individual (labeled by k) in generation n . Here, resource creation is supposed to summarise production of resources and non-consumption of resources originally claimed.

We make several assumptions, which are natural with respect to our interpretation:

(A1) D is an integer-valued non-negative random variable with law

$$p_k = P[D = k],$$

satisfying $p_0 > 0$ and, to avoid trivialities, satisfying also $p_0 + p_1 < 1$. Let m denote the mean reproduction rate, i.e. the expected number of offspring of one individual; we suppose

$$E[D] = \sum_{k=1}^{\infty} kp_k =: m < \infty.$$

(A2) R is a real-valued non-negative random variable. Let r denote the average resource creation; we suppose

$$0 < E[R] =: r < \infty.$$

(A3) X is an absolutely continuous real-valued non-negative random variable. Let F be the distribution function; we assume that the density $f = dF/dx$ has bounded support $[a, b]$. Let μ denote the average resource claim of one individual. Hence,

$$0 \leq E[X] =: \mu < \infty.$$

(A4) We shall assume throughout this paper that the random variables D , X and R are independent, so that $(D_n^k)_{n,k}$, $(X_n^k)_{n,k}$ and $(R_n^k)_{n,k}$ are three independent double arrays of i.i.d. random variables.

We of course realise that, in a convincing model, the random variables D , X and R should allow for some interaction (dependence), but assumption **(A4)** is made for simplicity.

Let now

$$D_n(k) := \sum_{j=1}^k D_n^j \quad \text{and} \quad R_n(k) := \sum_{j=1}^k R_n^j, \tag{1}$$

denote the total number of offspring and the total resources produced by generation n , respectively, given that generation n counts k individuals. The i.i.d. assumptions for random variables within the same double array allow us to use the shorter notations $D(k) = D_n(k)$ and $R(k) = R_n(k)$ whenever we limit our interest to their *distributional* prescriptions. Conversely, this is understood throughout the paper whenever we use this simplified notation.

In general, the total resources produced by a generation may be insufficient to satisfy all the resource claims of the offspring. We define a policy as a function which determines a priority order among offspring, i.e. a rule to distribute among the next generation the resources produced by the current generation.

Definition 3.1 (Global definition of a policy). A *policy* is any function of the form

$$\pi : \bigcup_{t \in \mathbb{N}} (\{t\} \times (\mathbb{R}^+)^t) \rightarrow \bigcup_{t \in \mathbb{N}} (\{t\} \times \text{Sym}(t)) : (t, (x_k)_{k=1}^t) \mapsto (t, \pi_t((x_k)_{k=1}^t)),$$

where $\text{Sym}(t)$ denotes the set of all permutations of $[t] := \{1, \dots, t\}$.

In this definition, t corresponds to the number of offspring, and $(x_k)_{k=1}^t$ to their respective resource claims (i.e. independent realisations of the random variable X). The permutation $\pi_t((x_k)_{k=1}^t) \in \text{Sym}(t)$ then gives the priority order that the society has chosen to satisfy the claims of the offspring: the individual $\pi_t((x_k)_{k=1}^t)(1)$ is the first served, etc. If s denotes the total of resources produced by the previous generation, the number of offspring having their claims completely satisfied thanks to the society's policy π is thus defined by

$$Q^\pi(t, (x_k)_{k=1}^t, s) = \begin{cases} 0, & \text{if } t = 0 \text{ or } x_{\pi_t((x_k)_{k=1}^t)(1)} > s, \\ \max \left\{ 1 \leq k \leq t : \sum_{j=1}^k x_{\pi_t((x_k)_{k=1}^t)(j)} \leq s \right\}, & \text{otherwise.} \end{cases}$$

Note that this function Q^π necessarily satisfies

$$Q^\pi(0, \emptyset, s) = 0 = Q^\pi(t, (x_k)_{k=1}^t, 0) \quad \text{and} \quad 0 \leq Q^\pi(t, (x_k)_{k=1}^t, s) \leq t,$$

for all $s \in \mathbb{R}^+$, all $t \in \mathbb{N}$ and all $(x_k)_{k=1}^t \in (\mathbb{R}^+)^t$.

We now recall (see Section 2) that all the offspring that are not completely satisfied, and only these, leave the society forever. This leads to the definition of the following stochastic process.

Definition 3.2 (Global model). If π is some policy, the *resource dependent branching process* (RDBP) on $(D_n^k, X_n^k, R_n^k)_{n,k}$ controlled by π is defined as the integer-valued non-negative stochastic process $(\Gamma_n)_{n \in \mathbb{N}}$, defined by $\Gamma_0 = 1$ and recursively

$$\Gamma_{n+1} = Q^\pi(D_n(\Gamma_n), (X_n^k)_{k=1}^{D_n(\Gamma_n)}, R_n(\Gamma_n)),$$

where $D_n(\cdot)$ and $R_n(\cdot)$ are given by Equation (1).

Such processes are Markov (see Proposition 5.1 hereafter).

A remark on multiparameter policies

According to our definition, a policy can only depend on the available resources and on the claims of the offspring. However, in more realistic models, the offspring could be characterised by many other different parameters, and it would be natural to allow a policy to depend on all these additional parameters. This is why, although we do not pursue such general models in this paper, we want to indicate shortly how we could adapt our definitions accordingly.

We thus consider a new double array $(\vec{Y}_n^k)_{n,k}$ of random p -vectors defined on a corresponding probability space (Ω, \mathcal{F}, P) , such that all quadruplets $(D_n^k, X_n^k, \vec{Y}_n^k, R_n^k)$ are i.i.d., with the same distribution as some quadruplet (D, X, \vec{Y}, R) . Here, the components of the random vectors \vec{Y}_n^k ($k \in \mathbb{N}_0$) correspond to the different characteristic parameters of each individual (labeled by k) in generation n .

Definition 3.3. A *p-parameter policy* is any function of the form

$$\pi : \bigcup_{t \in \mathbb{N}} (\{t\} \times (\mathbb{R}^+ \times B)^t) \rightarrow \bigcup_{t \in \mathbb{N}} (\{t\} \times \text{Sym}(t)) : (t, (x_k, \vec{y}_k)_{k=1}^t) \mapsto (t, \pi_t((x_k, \vec{y}_k)_{k=1}^t)),$$

where $p \geq 0$, $\vec{y}_k = (y_k^{(1)}, \dots, y_k^{(p)})$, and $B \subset \mathbb{R}^p$ the set of possible parameter values.

The associated counting function is thus defined by

$$Q^\pi(t, (x_k, \vec{y}_k)_{k=1}^t, s) = \begin{cases} 0, & \text{if } t = 0 \text{ or } x_{\pi_t((x_k, \vec{y}_k)_{k=1}^t)(1)} > s, \\ \max \left\{ 1 \leq k \leq t : \sum_{j=1}^k x_{\pi_t((x_k, \vec{y}_k)_{k=1}^t)(j)} \leq s \right\}, & \text{otherwise.} \end{cases}$$

Definition 3.4. If π is a p -parameter policy, the *RDBP* on $(X_n^k, \vec{Y}_n^k, D_n^k, R_n^k)_{k,n}$ controlled by π is defined as the integer-valued non-negative stochastic process $(\Gamma_n)_{n \in \mathbb{N}}$, defined by $\Gamma_0 = 1$ and recursively

$$\Gamma_{n+1} = Q^\pi(D_n(\Gamma_n), (X_n^k, \vec{Y}_n^k)_{k=1}^{D_n(\Gamma_n)}, R_n(\Gamma_n)).$$

Such processes are again Markov. As an example, we might think of some “Darwinian policy”, which gives the priority to the claims of the (in several respects) fittest individuals.

4 Particular policies

In the following, we define policies we think of as being of particular interest. The first will be a neutral policy, which we call the *first-come-first-served* policy. It will serve as a point of comparison with the *weakest-first* policy and the *strongest-first* policy defined later. The latter two policies turn out to be *extreme* in a sense to be clarified. The reader will notice that, in a first approximation, these two extreme policies have some important features in common with the principles of certain society forms we believe to know rather well.

4.1 First-come-first-served policy

The fcfs-policy is a neutral policy in the vein that it serves the claims according to their respective arrival times. To exclude ambiguities in the definition, these arrival times of claims are seen as lying at the beginning of each generation, being almost surely different, and all preceding the times of producing offspring.

Definition 4.1. The *first-come-first-served policy* (fcfs-policy) is the deterministic policy π^U defined by $\pi_t^U((x_k)_{k=1}^t) = \text{id}_{[t]}$.¹

The associated counting function $C := Q^{\pi^U}$ thus reads

$$C(t, (x_k)_{k=1}^t, s) = \begin{cases} 0, & \text{if } t = 0 \text{ or } x_1 > s \\ \sup \left\{ 1 \leq k \leq t : \sum_{j=1}^k x_j \leq s \right\}, & \text{otherwise.} \end{cases}$$

Definition 4.2. The *first-come-first-served process* (fcfs-process) on $(X_n^k, D_n^k, R_n^k)_{n,k}$ is the RDBP controlled by π^U , i.e. the stochastic process $(U_n)_{n \in \mathbb{N}}$ defined by $U_0 = 1$, and recursively by

$$U_{n+1} = C(D_n(U_n), (X_n^k)_{k=1}^{D_n(U_n)}, R_n(U_n)).$$

Note that $C(t, (X_n^k)_{k=1}^t, s)$ is a stopping time with respect to the natural filtration $(\mathcal{F}_\ell)_\ell$, where \mathcal{F}_ℓ denotes the σ -field generated by the X_n^k 's for $1 \leq k \leq \ell$.

Interpretation and properties. The fcfs-society may be seen as a model of a *laissez-faire* society. Individuals are born, arrive at (a.s.) different times within their generation at maturity and submit their random resource claims. This continues as long as resources are available. Since the claims are i.i.d. random variables, it is not the society but the scarcity of resources which may impose constraints. This process has some similarity with the GWP, because, for given distributions of resource production and claims, the claims curtail the (effective) mean m of the intrinsic offspring distribution $(p_k)_k$. This is why, among the special processes we consider, the fcfs-process has the easiest structure.

¹The notation π^U should remind of the unordered x_t^1, \dots, x_t^t used in the definition.

4.2 Weakest-first policy

The weakest-first policy (wf-policy) is an extreme policy, giving priority successively to the least demanding currently remaining offspring.

Definition 4.3. The *weakest-first policy* (wf-policy) is the deterministic policy π^W defined by $\pi_t^W((x_k)_{k=1}^t) = \sigma$, where σ is the permutation of $[t]$ such that $x_{\sigma(1)} \leq \dots \leq x_{\sigma(t)}$.

Throughout this paper, for i.i.d. realisations $(x_k)_{k=1}^t$ of the random variable X , the increasing order statistics will be denoted by $x_{1,t} \leq x_{2,t} \leq \dots \leq x_{t,t}$. The associated counting function $N := Q^{\pi^W}$ is now

$$N(t, (x_k)_{k=1}^t, s) = \begin{cases} 0, & \text{if } t = 0 \text{ or } x_{1,t} > s, \\ \sup \left\{ 1 \leq k \leq t : \sum_{j=1}^k x_{j,t} \leq s \right\}, & \text{otherwise.} \end{cases} \quad (2)$$

Definition 4.4. The *weakest-first process* (wf-process) on $(D_n^k, X_n^k, R_n^k)_{n,k}$ is the RDBP controlled by π^W , i.e. the stochastic process $(W_n)_{n \in \mathbb{N}}$ defined by $W_0 = 1$, and recursively by

$$W_{n+1} = N(D_n(W_n), (X_n^k)_{k=1}^{D_n(W_n)}, R_n(W_n)).$$

Note that $N(\cdot, \cdot, \cdot)$ counts the maximal number of increasing order statistics of the random sample $(x_k)_{k=1}^t$ which, starting with the smallest, can be summed up without exceeding s . Further, $N(t, (X_n^k)_{k=1}^t, s)$ is a stopping time on the filtration $(\mathcal{F}_\ell^I)_\ell$ say, generated by the ℓ first increasing order statistics from all order statistics, beginning with the smallest one, but it is not a stopping time with respect to the natural filtration $(\mathcal{F}_\ell)_\ell$.

Interpretation and properties. The policy of the wf-society is to support always the weakest. In that respect it comes close to the ideas of socialism and communism. In each generation, individuals are ordered according to their resource claims, and these order statistics are highly dependent of each other. Note that now a comparison with a GWP is (generation-wise) no longer possible.

4.3 Strongest-first policy

The strongest-first policy (sf-policy) is another extreme policy, giving successively priority to the most demanding currently remaining offspring.

Definition 4.5. The *strongest-first policy* (sf-policy) is the deterministic policy π^S defined by $\pi_t^S((x_k)_{k=1}^t) = \sigma$, where σ is the permutation of $[t]$ such that $x_{\sigma(1)} \geq \dots \geq x_{\sigma(t)}$.

The associated counting function $M := Q^{\pi^S}$ becomes

$$M(t, (x_k)_{k=1}^t, s) = \begin{cases} 0, & \text{if } t = 0 \text{ or } x_{t,t} > s \\ \sup \left\{ 1 \leq k \leq t : \sum_{j=t-k+1}^t x_{j,t} \leq s \right\}, & \text{otherwise,} \end{cases} \quad (3)$$

thus counting the maximal number of decreasing order statistics which can be summed up, starting with the biggest, without exceeding s .

Definition 4.6. The *strongest-first process* (sf-process) on $(X_n^k, D_n^k, R_n^k)_{n,k}$ is the RDBP controlled by π^S , i.e. the stochastic process $(S_n)_{n \in \mathbb{N}}$ defined by $S_0 = 1$, and recursively by

$$S_{n+1} = M(D_n(S_n), (X_n^k)_{k=1}^{D_n(S_n)}, R_n(S_n)).$$

We note that $M(t, (X_n^k)_{k=1}^t, s)$ is a stopping time on the filtration $(\mathcal{F}_\ell^D)_\ell$ generated by the first ℓ decreasing order statistics of all present claims, beginning with the largest one. It is again no stopping time on the natural filtration $(\mathcal{F}_\ell)_\ell$.

Interpretation and properties. The sf-society is the model which serves the strongest individuals first. Recall that we identified the values of resource claims with the power to defend these claims. Hence this society shares important features with free-market policies and an uncontrolled capitalistic society. Since claims are again highly dependent, the technical difficulty in this model is comparable with the one evoked for the wf-society. However, as we shall see, there are non-negligible differences.

5 Main Results

Before we present our results, we should, viewing motivation, draw attention to the fact that some of them are deeper than they might look.

Here is one example. Let $(\Gamma_n)_n$ be a process living under an arbitrary policy π . Since the sf-society is clearly the most restrictive one for the number of offspring which can stay, one feels that $(\Gamma_n)_n$ should always do at least as well as the process $(S_n)_n$ governed by the sf-policy. Now, if for instance sf-process $(S_n)_n$ and $(\Gamma_n)_n$ have the same number k of individuals at time n , then it follows from the definition that Γ_{n+1} is at least as large as S_{n+1} . Hence we expect in generation $n+2$ on average more offspring from Γ_{n+1} than from S_{n+1} . But then the extreme claims of the offspring of Γ_{n+1} may be much larger than those from the offspring of S_{n+1} so that the inequality may, in generation $n+2$, point to the opposite direction.

Throughout this Section, all RDBPs are supposed to be controlled by some policy π on some double array $(D_n^k, X_n^k, R_n^k)_{n,k}$ of i.i.d. triples of random variables satisfying assumptions **(A1)**-**(A4)** of Section 3.

We begin with the easier results.

It is important to first point out that any RDBP shares the following property which is so typical for most branching processes, namely either it explodes, or it becomes extinct.

Proposition 5.1. *Any RDBP $(\Gamma_n)_n$ is a Markov process. Moreover, it tends a.s. either to 0 or to ∞ . The same result remains true in the multiparameter case.*

Proof. See Section 7.1. □

In accordance with **Hypothesis 1** of Section 1, we must first answer the question under which conditions a given RDBP $(\Gamma_n)_n$ can survive, i.e. we must determine when the extinction probability

$$q_\Gamma = \mathbb{P} \left[\lim_{n \rightarrow \infty} \Gamma_n = 0 \mid \Gamma_0 = 1 \right]$$

is equal to 1, which will always be a central question in this paper. Note that, in the case when $q_\Gamma < 1$, the probability of extinction can intuitively be made arbitrary small if we replace the initial setting $\Gamma_0 = 1$ by $\Gamma_0 = M$ for M sufficiently large. This seems hard to prove in all generality. For the wf-process, however, this is true and a trivial consequence of the following stronger result:

Proposition 5.2. *For all $M > 0, M \in \mathbb{N}$,*

$$\mathbb{P} \left[\lim_{n \rightarrow \infty} W_n = 0 \mid W_0 = M \right] \leq q_W^M.$$

Proof. See Section 7.1. □

A similar result can easily be shown for the fcfs-process. Nevertheless, determining a precise upper bound for $\mathbb{P}[\lim_{n \rightarrow \infty} \Gamma_n = 0 \mid \Gamma_0 = M]$ which goes to 0 as M goes to infinity, remains an open question for the sf-process and general RDBPs.

5.1 Multiparameter policies

If the additional parameters of an individual are assumed to be independent of its number of offspring, its resource claim and its resource production, and if we consider some multiparameter policy that only depends on these additional parameters (but not on the resource claims), then the associated RDBP has exactly the same behaviour as the fcfs-process:

Proposition 5.3. *Assume that \vec{Y} is independent of D, X and R , and that the multiparameter policy π can be written under the form $\pi_t((x_k, \vec{y}_k)_{k=1}^t) = \rho((\vec{y}_k)_{k=1}^t)$. Let $(\Gamma_n)_n$ denote the associated RDBP on $(D_n^k, X_n^k, \vec{Y}_n^k, R_n^k)_{n,k}$. Then*

$$\mathbb{P}[\Gamma_n = m \mid \Gamma_0 = m_0] = \mathbb{P}[U_n = m \mid U_0 = m_0] \quad \forall n, m, m_0,$$

where $(U_n)_n$ is the corresponding fcfs-process. In particular, $q_\Gamma = q_U$.

Proof. See Section 7.2. □

In general, the dependence may of course lead to highly complex situations, which we shall not study in this paper. Nevertheless, the wf-policy and the sf-policy being extreme policies, they give relevant sharp bounds for all general cases.

5.2 General bounds

It turns out that the wf-process is always an upper bound for any other RDBP, in a very strong sense. More precisely,

Proposition 5.4. *Let $(\Gamma_n)_n$ be any RDBP on $(D_n^k, X_n^k, R_n^k)_{n,k}$, and let $(W_n)_n$ be the wf-process on the same double array. Then, for all n , we have $\Gamma_n \leq W_n$ a.s. In particular, $q_W \leq q_\Gamma$. The same result remains true in the multiparameter case.*

Proof. See Section 7.3. □

We now turn to a comparison between $(\Gamma_n)_n$ and the corresponding sf-process $(S_n)_n$, which is a much more subtle problem. Indeed, we have already pointed out that it is in general not true that $S_n \leq \Gamma_n$ a.s. for all n (see Section 7.3 for an explicit counterexample). Therefore all our attempts to compare trajectories turned out to be fruitless.

We found it highly interesting that, nevertheless, we can prove that

$$q_W = 1 \Rightarrow q_\Gamma = 1 \Rightarrow q_S = 1$$

(see Lemma 7.1 in Section 7.3), and remarkably, this suffices to deduce the following much stronger result:

Theorem 5.1. *Let $(\Gamma_n)_n$ be any RDBP on $(D_n^k, X_n^k, R_n^k)_{n,k}$, and let $(W_n)_n$ and $(S_n)_n$ be the wf-process and the sf-process on the same double array. Then,*

$$\lim_{n \rightarrow \infty} S_n \leq \lim_{n \rightarrow \infty} \Gamma_n \leq \lim_{n \rightarrow \infty} W_n \quad \text{a.s.}$$

In particular, $q_W \leq q_\Gamma \leq q_S$. The same results remain true in the multiparameter case.

Proof. See Section 7.3. □

Such bounds are of considerable theoretical interest, and, as we shall now see, they are also serving as useful directives for individuals who have decided to adapt a specific type of society. Indeed, if the probability laws of the random variables $(D_n^k)_{n,k}$, $(X_n^k)_{n,k}$ and $(R_n^k)_{n,k}$ are fixed up to their mean m , μ and r , respectively, then it is in practice interesting to determine the *critical mean resource production* $r_{\Gamma,c}(m, \mu)$, say, relative to the RDBP $(\Gamma_n)_n$, i.e. the value such that

$$q_\Gamma = 1 \quad \text{if } r < r_{\Gamma,c}(m, \mu) \quad \text{and} \quad q_\Gamma < 1 \quad \text{if } r > r_{\Gamma,c}(m, \mu).$$

By Theorem 5.1, the following can be deduced:

Corollary 5.1. *For all m, μ , we have*

$$r_{W,c}(m, \mu) \leq r_{\Gamma,c}(m, \mu) \leq r_{S,c}(m, \mu).$$

Therefore, the study of the two extreme RDBPs gives non-trivial information about general RDBPs, without having to understand every single possible policy (see examples in Section 6). In the sequel, we shall thus essentially restrict our attention to the wf-policy and the sf-policy, as well as to the neutral fcfs-policy as a point of comparison.

As we shall see in the next subsections, the computation of the critical mean resource production even shows more. The point is that the mean claim value plays only one part but that the resource claim distribution function F (which determines the mean, of course) plays itself an important part. Hence society may try to take influence on individuals to settle, under a fixed mean claim μ , for a distribution F which favours survival.

Remark 5.1. If D , X and R were not assumed to be independent, Theorem 5.1 would in general not remain true: the wf-policy and the sf-policy would *a priori* not remain extreme policies. We could then naturally wonder how different dependence patterns yield different extreme policies. Such questions may attract interest for further studies.

5.3 The wf-process

Theorem 5.2. Let $(W_n)_n$ be the wf-process on $(D_n^k, X_n^k, R_n^k)_{n,k}$, where this double array of i.i.d. triples of random variables satisfies assumptions **(A1)**-**(A4)** of Section 3. Suppose $m > 1$ and $\mu > 0$.

(a) If $r \leq m\mu$ and if τ is the solution of

$$\int_0^\tau x \, dF(x) = \frac{r}{m}, \quad (4)$$

then

- (i) if $mF(\tau) < 1$, then $q_W = 1$;
- (ii) if $mF(\tau) > 1$ and $F(r/k) > 0$ for some $k \geq 2$ with $p_k > 0$, then $q_W < 1$.

(b) If $r > m\mu$ and $F(r/k) > 0$ for some $k \geq 2$ with $p_k > 0$, then $q_W < 1$.

Further, if there is no extinction, the process explodes a.s. and behaves asymptotically like a supercritical GWP with a new reproduction mean $\tilde{m}(> 1)$, say, defined by

$$\tilde{m} = \begin{cases} m, & \text{if } r \geq m\mu, \\ mF(\tau), & \text{if } r < m\mu \text{ and } mF(\tau) > 1. \end{cases}$$

Proof. See Section 7.4. □

The following remarks will provide a better understanding of these results.

Remarks 5.1.

- i) We have assumed here that the resource claims are bounded (recall **A3**). However, we do not use this assumption in the proof of the above result.
- ii) The case $m \leq 1$ is trivial because then $(W_n)_n$ is stochastically smaller than a subcritical GWP and therefore bound to die out, since $p_0 > 0$. Further, since all resource claims are non-negative, the case $\mu = 0$ is trivial too, because, if $(X_n^k)_{n,k}$ consists only of 0's, then the process coincide with the standard GWP: in this case, survival is possible if and only if $m > 1$.
- iii) Among the non-trivial cases, the case (b) is the most intuitive one. Indeed, the condition $r > m\mu$ means that a typical ancestor produces in expectation more resources than his offspring will claim together. Consequently, when the population grows the law of large numbers ensures that the process will finally behave like a supercritical GWP, the asymptotic properties of which are well understood; see Bingham and Doney [1974]. The additional condition $p_0 + p_1 < 1$ implies for $m > 1$ that $p_k > 0$ for some $k \geq 2$ so that the process can trigger off and reach any size with positive probability. This is a necessary condition for the preceding argument to hold; this condition becomes redundant if we replace the initial setting $W_0 = 1$ by $W_0 = w$ for w sufficiently large.
- iv) Theorem 5.2 is sharp in the sense that $mF(\tau) = 1$ is the exact separation point between a.s. extinction and positive survival probability. However, unlike what occurs with GWPs, it is here not immediate to see under which conditions on the law $\{p_k\}_k$ and F the critical case implies a.s. extinction. Note that, for fixed m and F , the parameter $\tau = \tau(r/m)$ is increasing in r , so that the equation $mF(\tau) = 1$ defines a critical mean resource production $r_{W,c}$ – below which $q_W = 1$ and above which $q_W < 1$ provided that the additional condition involving $F(r/k) > 0$ is satisfied.

We can further prove the following intuitive result.

Corollary 5.2. *Let $(W_n)_n$ be the wf-process on $(D_n^k, X_n^k, R_n^k)_{n,k}$, where this double array of i.i.d. triples of random variables satisfies assumptions **(A1)**-**(A4)** of Section 3. Assume $m > 1$, $\mu > 0$, and $F(r/k) > 0$ for some $k \geq 2$ with $p_k > 0$. Then, if $\mu < r$, we have $q_W < 1$.*

Proof. See Section 7.5. □

5.4 The sf-process

We now present the extinction criterion for the sf-process. Since we deal here again with a process depending on the partial sum behaviour of order statistics – now on the sum of the largest ones – we expect analogies. Since 0 is a lower bound for resource claims, the smallest order statistics are naturally bounded below. To facilitate a comparison between the sf-process and the wf-process we had made the assumption (recall **A3**) that resource claims are bounded above, i.e. that F has a finite support. This assumption facilitates the mathematical treatment, but is also highly reasonable as far as the interpretation of the model.

Theorem 5.3. *Let $(S_n)_n$ be the sf-process on $(D_n^k, X_n^k, R_n^k)_{n,k}$, where this double array of i.i.d. triples of random variables satisfies assumptions **(A1)**-**(A4)** of Section 3. Suppose $m > 1$ and $\mu > 0$.*

(a) *If $r \leq m\mu$ and if θ is the solution of*

$$\int_{\theta}^b x \, dF(x) = \frac{r}{m}, \quad (5)$$

then

- (i) *if $m(1 - F(\theta)) < 1$, then $q_S = 1$;*
- (ii) *if $m(1 - F(\theta)) > 1$ and $F(r/k) > 0$ for some $k \geq 2$ with $p_k > 0$, then $q_S < 1$.*

(b) *If $r > m\mu$ and $F(r/k) > 0$ for some $k \geq 2$ with $p_k > 0$, then $q_S < 1$.*

Further, if there is no extinction, the process explodes a.s. and behaves asymptotically like a supercritical GWP with a new reproduction mean $\tilde{m}(> 1)$, say, defined by

$$\tilde{m} = \begin{cases} m, & \text{if } r \geq m\mu, \\ m(1 - F(\theta)), & \text{if } r < m\mu \text{ and } m(1 - F(\theta)) > 1. \end{cases}$$

Proof. See Section 7.4. □

Remarks 5.2.

- i) As noted for wf-processes, the case $m \leq 1$ is trivial, implying $q_S = 1$. The case $\mu = 0$ is trivial too: the process then coincides with the standard GWP, so that survival is possible if and only if $m > 1$.
- ii) Again, the additional condition involving $F(r/k) > 0$ just serves to ensure that the process can grow. It becomes superfluous if we replace the initial setting $S_0 = 1$ by $S_0 = s$ for some s sufficiently large.
- iii) The critical case is now determined by the equation $m(1 - F(\theta)) = 1$. Note that, for fixed m , the parameter $\theta = \theta(r/m)$ is decreasing in r , in the same way that, in Theorem (5.2), $\tau(r/m)$ was increasing in r . Again, the equation $m(1 - F(\theta)) = 1$ thus defines the critical mean resource production $r_{S,c}$.

We have, and here intuition is correct, the following result.

Corollary 5.3. *Let $(S_n)_n$ be the sf-process on $(D_n^k, X_n^k, R_n^k)_{n,k}$, where, again, this double array of i.i.d. triples of random variables satisfies assumptions **(A1)**-**(A4)** of Section 3. Assume $m > 1$, $\mu > 0$, and F having finite support $[a, b]$, $0 \leq a < b < \infty$. Then, if $r < \mu$, we have $q_S = 1$.*

Proof. See Section 7.5. □

5.5 The fcfs-process

As a term of comparison, it is interesting to observe what happens in the case of a fcfs-process.

Proposition 5.5. *Let $(U_n)_n$ be the fcfs-process on $(D_n^k, X_n^k, R_n^k)_{n,k}$, where this double array of i.i.d. triples of random variables satisfies assumptions **(A1)**–**(A4)** of Section 3. Suppose $m > 1$ and $\mu > 0$.*

- (a) *If $r < \mu$, then $q_U = 1$.*
- (b) *If $r > \mu$, then $q_U < 1$.*

Further, if there is no extinction, the process explodes a.s. and behaves asymptotically like a super-critical GWP with reproduction mean m .

Proof. This follows immediately from the well-known properties of the GWP and the strong law of large numbers. \square

Remarks 5.3.

- i) We have assumed here that the resource claims are bounded (recall **A3**). However, we do not use this assumption in the proof of the above result.
- ii) As noted for wf-processes, the case $m \leq 1$ is trivial, implying $q_U = 1$. The case $\mu = 0$ is trivial too: the process then coincides with the standard GWP, so that survival is possible if and only if $m > 1$.
- iii) The critical mean resource production is now simply defined by $r_{U,c} = \mu$.

6 Examples

We now give examples. It will be interesting to notice that the critical mean resource production for a wf-process turns out to be lower than one would intuitively expect, and conversely for a sf-process.

- i) Let F be the uniform distribution function on $(0, d)$, say. Then $\mu = d/2$. As in Theorems 5.2(a) and 5.3(a), let $r < m\mu = md/2$ and suppose that $F(r/k) > 0$ for some $k \geq 2$ with $p_k > 0$. First, focus on the corresponding wf-process. The value of τ (see Equation (4)) is thus determined by:

$$\frac{r}{m} = \int_0^\tau x dF(x) = \int_0^\tau \frac{1}{d} x dx = \frac{\tau^2}{2d},$$

which yields $\tau = \sqrt{2dr/m}$. Therefore, $F(\tau) = \sqrt{2r/(md)}$. The critical mean resource production $r_{w,c}$ is thus determined by

$$mF(\tau) = 1 \quad \Leftrightarrow \quad \sqrt{2r_{w,c}m/d} = 1$$

which implies $r_{w,c} = d/2m = \mu/m$. Note that, for larger m , $r_{w,c}$ is quite close to the smallest or second smallest (expected) claim of offspring of two families. Indeed, $r_{w,c} = d/2m > d/(2m+1)$, which is the expected value of the smallest order statistic of $2m$ i.i.d. $U[0, d]$ random variables, and $r_{w,c} < 2d/(2m+1)$.

With such a low productivity of resources, the sf-process and the fcfs-process would die out very quickly, as we shall see now.

In the case of a sf-process, we need to determine θ , which is defined by (see Equation (5)):

$$\frac{r}{m} = \int_\theta^d \frac{1}{d} x dx = \frac{d^2 - \theta^2}{2d}$$

and so, straightforward calculations yield $r_{s,c} = d(1 - 1/2m) = \mu(2 - 1/m)$.

We note that the critical mean resource production is now $2m - 1$ times higher than for the

corresponding wf-process. Moreover, comparing with the corresponding critical mean resource production for a fcfs-process, $r_{u,c} = \mu$, gives

$$r_{u,c} - r_{w,c} = \mu(1 - 1/m) = r_{s,c} - r_{u,c}.$$

Figure 1 compare the behaviour of $r_{w,c}$ and $r_{s,c}$ as functions of m . The area between the two curves corresponds to a *critical area*, where the population can both survive or die, depending on the policy. This shows how the study of the two extreme RDBPs yields highly non-trivial information about general RDBPs, without having to understand every single possible policy, as already pointed out in Section 5.2.

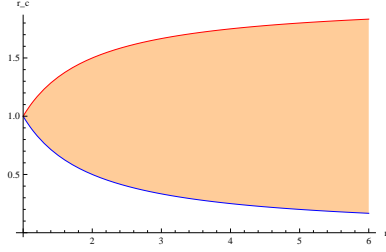


Figure 1: For $\mu = 1$, the critical mean resource productions $r_{w,c}$ and $r_{s,c}$ are plotted (in blue and in red, respectively), as functions of m .

- ii) Of course, we realise that the uniform distribution pushes the largest and the smallest order statistics far apart, namely, for n observations, by a factor n (in expectation). Therefore, we find it informative to look at the case when the resource claim distribution is more concentrated around its mean, as for instance in the case of a beta distribution (on $(0, 1)$, with parameters a and b).

The distribution function is then defined on $(0, 1)$ by the regularised incomplete beta function: $F(x) = I_{a,b}(x)$. The mean resource claim is given by $\mu = \frac{a}{a+b}$. As in Theorems 5.2(a) and 5.3(a), let $r < m\mu = \frac{am}{a+b}$ and suppose that $F(r/k) > 0$ for some $k \geq 2$ with $p_k > 0$.

First, focus on the corresponding wf-process. The value of τ (see Equation (4)) is determined by:

$$\frac{r}{m} = \int_0^\tau x dF(x) = \int_0^\tau \frac{x^a(1-x)^{b-1}}{B(a,b)} = \frac{B(a+1,b)}{B(a,b)} I_{a+1,b}(\tau) = \frac{a}{a+b} I_{a+1,b}(\tau),$$

which yields $\tau = I_{a+1,b}^{-1}\left(\frac{r}{m} \frac{a+b}{a}\right)$. The critical mean resource production $r_{w,c}$ is thus determined by

$$mF(\tau) = 1 \quad \Leftrightarrow \quad m I_{a,b}\left(I_{a+1,b}^{-1}\left(r_{w,c} \frac{a+b}{am}\right)\right) = 1,$$

which implies

$$r_{w,c} = \frac{am}{a+b} I_{a+1,b}(I_{a,b}^{-1}(1/m)).$$

Now look at the corresponding sf-process. The value of θ (see Equation (5)) is determined by:

$$\frac{r}{m} = \int_\theta^1 x dF(x) = \frac{a}{a+b} (1 - I_{a+1,b}(\theta)),$$

which yields $\theta = I_{a+1,b}^{-1}\left(1 - \frac{r}{m} \frac{a+b}{a}\right)$. Straightforward calculations then give the critical mean resource production $r_{s,c} = \frac{am}{a+b} (1 - I_{a+1,b}(I_{a,b}^{-1}(1 - 1/m)))$.

Observing that $I_{\alpha,\beta}(1-x) = 1 - I_{\beta,\alpha}(x)$, we deduce that $I_{\alpha,\beta}^{-1}(1-x) = 1 - I_{\beta,\alpha}^{-1}(x)$. The formula for $r_{s,c}$ can thus be rewritten as

$$r_{s,c} = \frac{am}{a+b} I_{b,a+1} \left(I_{b,a}^{-1}(1/m) \right).$$

For a fcfs-process, the corresponding critical mean resource production simply reads $r_{u,c} = \mu$. Further, for big m , we could use the approximation

$$I_{\alpha,\beta}(z) = \frac{z^\alpha}{B(\alpha,\beta)} \left(\frac{1}{\alpha} + \frac{1-\beta}{\alpha+1}z + O(z^2) \right)$$

(see, e.g., [Pearson \[1968\]](#)). Straightforward calculations then give, at leading order:

$$r_{w,c} = \frac{a}{a+1} \left(\frac{a}{m} B(a,b) \right)^{1/a} + O(m^{-2/a}),$$

and

$$r_{s,c} = 1 + \frac{b}{b+1} (b - a(b+1)) \left(\frac{b}{m} B(b,a) \right)^{1/b} + O(m^{-2/b}).$$

Figure 2 shows the critical areas in some typical cases, as the peak is centered, moved to the left or to the right. Figure 3 shows, in the centered case, how the critical area narrows as the dispersion around the peak diminishes.

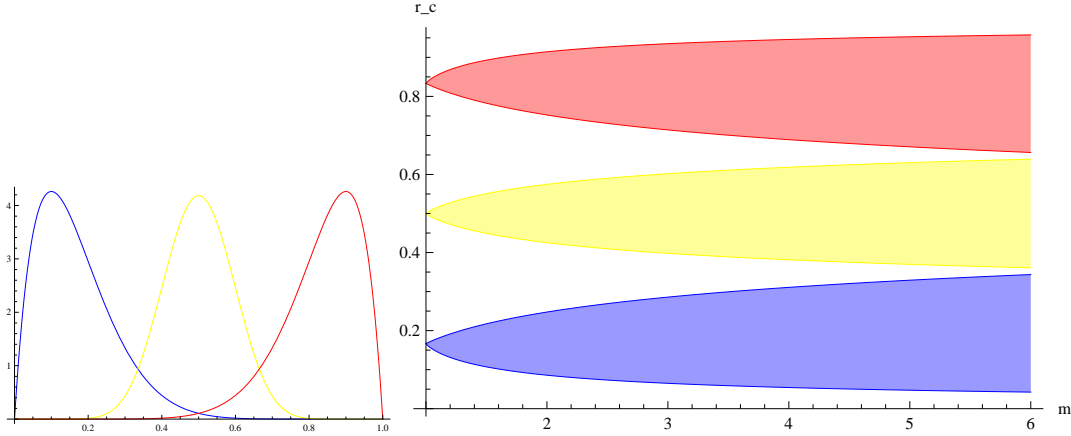


Figure 2: The critical mean resource productions $r_{w,c}$ and $r_{s,c}$ are plotted as functions of m , in the case of a $B(a,b)$ resource claim distribution, for typical values of (a,b) : $(2,10)$ in blue, $(14,14)$ in yellow, and $(10,2)$ in red.

- iii) Let F be the distribution function of an exponential random variable with parameter λ . This is not bounded and our results in the sf-case cannot be used. However, it could be interesting to see what happens for the corresponding wf-process. The mean resource claim is given by $\mu = 1/\lambda$. As in Theorem 5.2(a), let $r < m\mu = m/\lambda$ and suppose that $F(r/k) > 0$ for some $k \geq 2$ with $p_k > 0$.

The value of τ is determined by (see Equation (4))

$$\frac{r}{m} = \int_0^\tau x dF(x) = \int_0^\tau \lambda x e^{-\lambda x} dx = \frac{1}{\lambda} - e \left(\tau + \frac{1}{\lambda} \right) e^{-\lambda(\tau+1/\lambda)}$$

which yields $\tau = -\frac{1}{\lambda} \left(1 + W \left[-\frac{\lambda}{e} \left(\frac{1}{\lambda} - \frac{r}{m} \right) \right] \right)$, where $W[\cdot]$ denotes the Lambert W function (see, e.g., [Corless, Gonnet, Hare, Jeffrey, and Knuth \[1996\]](#)). The critical mean resource production,

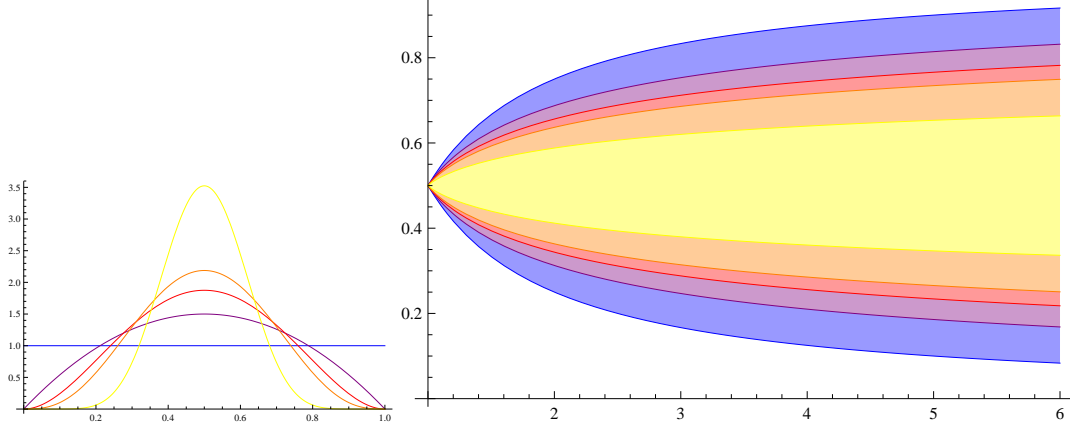


Figure 3: The critical mean resource productions $r_{w,c}$ and $r_{s,c}$ are plotted as functions of m , in the case of a $B(a, b)$ resource claim distribution, for different symmetric values of (a, b) : $(1, 1)$ in blue, $(2, 2)$ in pink, $(3, 3)$ in red, $(4, 4)$ in orange and $(10, 10)$ in red.

$r_{w,c}$, is thus determined by

$$mF(\tau) = 1 \quad \Leftrightarrow \quad m - e m \exp \left(W \left[-\frac{\lambda}{e} \left(\frac{1}{\lambda} - \frac{r_{w,c}}{m} \right) \right] \right) = 1.$$

After simplifications, we get $r_{w,c} = \frac{1}{\lambda} \left(1 - (m-1) \log \left(\frac{m}{m-1} \right) \right)$, where we recall that $\frac{1}{\lambda} = \mu$. For big m , this becomes $r_{w,c} \approx 1/\lambda m = \mu/m$.

7 Proofs

7.1 Preliminary results

We prove here Propositions 5.1 and 5.2.

Proof of Proposition 5.1. Assume π to be some policy. Given Γ_n , the distributions of $D_n(\Gamma_n)$ and $R_n(\Gamma_n)$ are independent from $\Gamma_1, \dots, \Gamma_{n-1}$, so that $\Gamma_{n+1} = Q^\pi(D_n(\Gamma_n), (X_n^k)_{k=1}^{D_n(\Gamma_n)}, R_n(\Gamma_n))$ is independent of $\Gamma_1, \dots, \Gamma_{n-1}$, given Γ_n . Thus, $(\Gamma_n)_n$ is a Markov process.

Now note that, since $Q^\pi(0, \emptyset, s) = 0$ and $D_n(0) = 0$ for all $n \in \mathbb{N}$, we have $\{\Gamma_n = 0\} \subset \{\Gamma_{n+1} = 0\}$ so that 0 is an absorbing state for the process $(\Gamma_n)_n$. Moreover, since

$$\Gamma_{n+1} = Q^\pi(D_n(\Gamma_n), (X_n^k)_{k=1}^{D_n(\Gamma_n)}, R_n(\Gamma_n)) \leq D_n(\Gamma_n), \quad (6)$$

it follows that

$$P[\Gamma_{n+1} = 0 | \Gamma_n] \geq P[D_n(\Gamma_n) = 0 | \Gamma_n] = p_0^{\Gamma_n}, \quad (7)$$

where the last equality holds because of the assumption of independent reproduction. Therefore, the absorbing state 0 is accessible from any state $s \in \mathbb{N}$, with at least probability $p_0^s > 0$. The state 0 is thus the only absorbing state, and, as $(\Gamma_n)_n$ is a Markov process, we conclude

$$P[0 < \Gamma_n \leq s \text{ i.o.}] = 0, \quad \forall s \in \mathbb{N}_0. \quad (8)$$

The same arguments immediately adapt to multiparameter policies. \square

We now turn to the proof of Proposition 5.2.

Proof of Proposition 5.2. Let $M \in \mathbb{N}_0$ and let $W_n^{(1)}, \dots, W_n^{(M)}$ be M i.i.d. copies of a weakest-first process. We then have the following (superadditivity-type) inequality, namely, for all $n, k \in \mathbb{N}_0$,

$$P[W_n \leq k | W_0 = M] \leq P[W_n^{(1)} + \dots + W_n^{(M)} \leq k | W_0^{(1)} = \dots = W_0^{(M)} = 1], \quad (9)$$

which we shall prove first.

To see this, we begin with the case $n = 1$. The lhs of (9) becomes, by an additional conditioning on $D^1(M) = D(M)$,

$$P[W_1 \leq k | W_0 = M] = P[D(M) \leq k] + P[W_1 \leq k | D(M) > k, W_0 = M]P[D(M) > k],$$

since the distribution of $D(M)$ depends only on M , and the probability that W_1 does not exceed $D(W_0) = D(M)$ equals 1.

Now suppose we do the same conditioning on the rhs of (9), that is, for the offspring of the M partitioned processes. The first term $P[D(M) \leq k]$ is the same on both sides, as reproduction is independent. Hence we can limit our interest to the corresponding second term with more than k offspring.

The distributions of the total created resource space and of the claims are also by definition the same on both sides; therefore it suffices to look for the moment at the influence of the order statistics of claims in a fixed sequence of claims on a fixed resource space R , say. In the lhs-model the resource space R is global (i.e., united) whereas in the rhs-model it is local (i.e., compartmented). On the lhs the count of individuals to stay is the count of the globally smallest order statistics of claims which can be accommodated by R whereas on the rhs the count is on the locally smallest order statistics of claims. The latter, put in increasing order, are a subsequence of the sequence of claims in increasing order. Hence the rhs count cannot exceed the lhs count. Passing from the count to the corresponding probability measures on both sides proves (9) for $n = 1$, that is, W_1 is stochastically larger than $W_1^{(1)} + \dots + W_1^{(M)}$.

But now, the inequality (9) must hold in particular if we replace on the rhs the number M by some M' with $M' \leq M$. Hence the stochastic order is maintained through the next generation, and thus, by recurrence, through all generations. This implies that (9) is true for all $n \in \mathbb{N}_0$.

Finally, choosing $k = 0$ in (9) and taking the limit for $n \rightarrow \infty$, we obtain by independence of the processes $(W_n^{(j)})_n$ that

$$P[W_n \rightarrow 0 | W_0 = M] \leq P[W_n^{(1)} \rightarrow 0, \dots, W_n^{(M)} \rightarrow 0 | W_0^{(1)} = \dots = W_0^{(M)} = 1] = q_W^M, \quad (10)$$

which completes the proof. \square

Remark: The superadditivity-type inequality (9) is no more correct if the wf-process is replaced by the sf-process. Indeed, a very large claim may now force on the rhs all the offspring of one subpopulation to leave, but this effect stays still local whereas it may be large on the global lhs. The same argument implies that the corresponding inequality of (9) is no longer true even for the fcs-process. This exemplifies at the same time the adherent difficulty in estimating extinction probabilities for arbitrary policies.

7.2 Multiparameter policies

We now prove Proposition 5.3.

Proof of Proposition 5.3. We prove by induction on n that

$$P[\Gamma_n = m | \Gamma_0 = m_0] = P[U_n = m | U_0 = m_0]$$

for all n, m, m_0 . It is clear for $n = 0$. Moreover, if it is true for some n , then we can write

$$\begin{aligned}
& P[\Gamma_{n+1} = m | \Gamma_0 = m_0] \\
&= \sum_{k,t=0}^{\infty} \sum_{\sigma \in \text{Sym}(t)} \int_0^{\infty} \dots \int_0^{\infty} dF(x_1) \dots dF(x_t) P[\rho_t((\vec{Y}_k)_{k=1}^t) = \sigma] P[\Gamma_n = k | \Gamma_0 = m_0] P[D_n(k) = t] \\
&\quad P[\Gamma_{n+1} = m | \Gamma_0 = m_0, \Gamma_n = k, D_n(k) = t, X_n^1 = x_1, \dots, X_n^t = x_t, \rho_t((\vec{Y}_k)_{k=1}^t) = \sigma] \\
&= \sum_{k,t=0}^{\infty} \sum_{\sigma \in \text{Sym}(t)} \int_0^{\infty} \dots \int_0^{\infty} dF(x_1) \dots dF(x_t) \frac{1}{t!} P[U_n = k | U_0 = m_0] P[D_n(k) = t] \\
&\quad P[U_{n+1} = m | U_0 = m_0, U_n = k, D_n(k) = t, X_n^1 = x_{\sigma(1)}, \dots, X_n^t = x_{\sigma(t)}] \\
&= \sum_{k,t=0}^{\infty} P[U_n = k | U_0 = m_0] P[D_n(k) = t] P[U_{n+1} = m | U_0 = m_0, U_n = k, D_n(k) = t] \\
&= P[U_{n+1} = m | U_0 = m_0],
\end{aligned}$$

and thus the statement is proved. \square

7.3 General bounds

We first prove Proposition 5.4.

Proof of Proposition 5.4. Let π be any policy (the same arguments immediately adapt to multiparameter policies). First note that, by definition of $N(\cdot)$ and $M(\cdot)$,

$$M(t, (x_k)_{k=1}^t, s) \leq Q^\pi(t, (x_k)_{k=1}^t, s) \leq N(t, (x_k)_{k=1}^t, s), \quad \forall t, \forall (x_k)_{k=1}^t, \forall s. \quad (11)$$

We shall now show by induction that if $(\Gamma_n)_n$ is the RDBP controlled by π , it follows that $\Gamma_n \leq W_n$ a.s. for all n , given $W_0 = \Gamma_0 = S_0 = 1$. Indeed, it is true at any time at which the three processes have the same number of individuals, and hence for $n = 0$. Now, if it is true for some n , we deduce that, a.s.,

$$\Gamma_{n+1} = Q^\pi(D_n(\Gamma_n), (X_n^k)_{k=1}^{D_n(\Gamma_n)}, R_n(\Gamma_n)) \quad (12)$$

$$\leq N(D_n(\Gamma_n), (X_n^k)_{k=1}^{D_n(\Gamma_n)}, R_n(\Gamma_n)) \quad (13)$$

$$\leq N(D_n(W_n), (X_n^k)_{k=1}^{D_n(W_n)}, R_n(W_n)) = W_{n+1}, \quad (14)$$

as the mapping $(t, s) \mapsto N(t, (x_k)_{k=1}^t, s)$ is increasing in both arguments. Hence the inequality is also true for $n + 1$.

It follows that $P(\Gamma_n \leq W_n) = 1$ for all n . Since the limiting extinction probabilities of $(\Gamma_n)_n$ and $(W_n)_n$ must exist, we must also have $q_W \leq q_\Gamma$. \square

We now give an explicit counterexample showing that it is in general not true that $S_n \leq \Gamma_n$ a.s. for all n , given $S_0 = \Gamma_0 = 1$. The underlying idea was already explained at the very beginning of Section 5.

Counterexample. We have assumed $p_k > 0$ for some $k \geq 2$ (see **(A1)**); to fix ideas, assume that $p_3 > 0$ (the argument can be adapted in any case). Then consider the deterministic policy π given by

$$\pi_t((x_k)_{k=1}^t)(j) = \begin{cases} \sigma(3), & \text{if } j = 1 \text{ and } t \geq 3, \\ \sigma(1), & \text{if } j = 2 \text{ and } t \geq 3, \\ \sigma(2), & \text{if } j = 3 \text{ and } t \geq 3, \\ \sigma(j), & \text{otherwise.} \end{cases} \quad (15)$$

where σ is the permutation such that $x_{\sigma(1)} \geq \dots \geq x_{\sigma(t)}$ (i.e., by definition, $\sigma = \pi_t^S((x_k)_{k=1}^t)$). Let for example

$$\begin{aligned} D_0^1 &= 3, X_0^1 > X_0^2 > X_0^3, X_0^1 + X_0^3 < R_0^1 < X_0^1 + X_0^2 \\ D_1^1 &= D_1^2 = 3, X_1^1 + X_1^2 + X_1^3 \leq R_1^1 \end{aligned}$$

and then

$$X_1^4, X_1^5, X_1^6 > R_1^1 + R_1^2.$$

These events will occur simultaneously with positive probability, as $p_3 > 0$. But then we immediately see that $\Gamma_2 = 0 < 3 = S_2$ in this case.

We now turn to the proof of Theorem 5.1. For that purpose, we shall need the following interesting result, which we prove first.

Theorem 7.1. *Let $(\Gamma_n)_n$ be any RDBP controlled by some policy π on $(D_n^k, X_n^k, R_n^k)_{n,k}$, where this double array of i.i.d. triples of random variables satisfies assumptions (A1)-(A4) of Section 3. Let $1 > \eta > 0$. For all $\delta > 0$, there exists some $L_\delta \in \mathbb{N}_0$ such that*

$$\mathbb{P} \left[\frac{\Gamma_{n+1}}{\Gamma_n} \geq \frac{S_{n+1}}{S_n} - \delta \mid \Gamma_n \geq L_\delta, S_n \geq L_\delta \right] \geq 1 - \eta.$$

The same result remains true in the multiparameter case.

Proof. Let $1 > \eta > 0$. Since $\frac{1}{k}D(k) \rightarrow m$ a.s. and $\frac{1}{k}R(k) \rightarrow r$ a.s., we have, for all $\epsilon > 0$,

$$\mathbb{P} \left[\sup_{k \geq m} \left| \frac{1}{k}D(k) - m \right| < \epsilon \right] \rightarrow 1 \quad \text{and} \quad \mathbb{P} \left[\sup_{k \geq m} \left| \frac{1}{k}R(k) - r \right| < \epsilon \right] \rightarrow 1, \quad (16)$$

as $m \rightarrow \infty$. Therefore, for all $\epsilon > 0$, we can find $K_\epsilon \in \mathbb{N}_0$ such that

$$(m - \epsilon)k \leq D(k) \leq (m + \epsilon)k \quad \text{and} \quad (r - \epsilon)k \leq R(k) \leq (r + \epsilon)k \quad (17)$$

must hold (simultaneously) for all $k \geq K_\epsilon$, with at least probability $1 - \eta/2$.

On the one hand, if we assume that $\sum_{k=1}^{D(k)} X_n^k \geq R(k)$ is given, we necessarily have, given $S_n = k$,

$$M(\lfloor (m - \epsilon)k \rfloor, (X_n^k)_{k=1}^{\lfloor (m - \epsilon)k \rfloor}, R(k)) \leq S_{n+1} \leq M(\lceil (m + \epsilon)k \rceil, (X_n^k)_{k=1}^{\lceil (m + \epsilon)k \rceil}, R(k)), \quad (18)$$

where $\lfloor x \rfloor$ denotes the largest integer $\leq x$ and where $\lceil x \rceil$ denotes the smallest integer $\geq x$. Since $M(t, (X_n^k)_{k=1}^t, s)$ is stochastically increasing in s , we thus have

$$M(\lfloor (m - \epsilon)k \rfloor, (X_n^k)_{k=1}^{\lfloor (m - \epsilon)k \rfloor}, (r - \epsilon)k) \leq S_{n+1} \leq M(\lceil (m + \epsilon)k \rceil, (X_n^k)_{k=1}^{\lceil (m + \epsilon)k \rceil}, (r + \epsilon)k) \quad (19)$$

for all $k \geq K_\epsilon$, with at least probability $1 - \eta/2$. Now, using this and Theorem 2.3 of [Bruss and Robertson \[1991\]](#), we deduce that, for all $\delta > 0$, there exists some $K'_{\delta, \epsilon} \geq K_\epsilon$ such that, for all $k \geq K'_{\delta, \epsilon}$,

$$\lfloor (m - \epsilon)k \rfloor \left(1 - F \left(\theta \left(\frac{(r - \epsilon)k}{\lfloor (m - \epsilon)k \rfloor} \right) \right) - \delta/(2m) \right) \quad (20)$$

$$\leq S_{n+1} \leq \lceil (m + \epsilon)k \rceil \left(1 - F \left(\theta \left(\frac{(r + \epsilon)k}{\lceil (m + \epsilon)k \rceil} \right) \right) + \delta/(2m) \right) \quad (21)$$

with at least probability $(1 - \eta/2)^2 \geq 1 - \eta$, where the continuous function $\theta(\cdot)$ is defined by

$$\int_{\theta(\rho)}^b x dF(x) = \rho. \quad (22)$$

Taking $\epsilon = \epsilon_\delta > 0$ small enough yields on the one hand

$$mk(1 - F(\theta(r/m)) - \delta/m) \leq S_{n+1} \leq mk(1 - F(\theta(r/m)) + \delta/m) \quad (23)$$

with at least probability $1 - \eta$, for all $k \geq K'_{\delta, \epsilon_\delta} =: K'_\delta$.

On the other hand, if we assume that $\sum_{k=1}^{D(k)} X_n^k < R(k)$ is given, then, conditioned on the event $\{S_n = k\}$, we must have

$$M(\lceil (m + \epsilon)k \rceil, (X_n^k)_{k=1}^{\lceil (m + \epsilon)k \rceil}, R(k)) \leq S_{n+1} \leq M(\lfloor (m + \epsilon)k \rfloor, (X_n^k)_{k=1}^{\lfloor (m + \epsilon)k \rfloor}, R(k)) \quad (24)$$

with at least probability $1 - \eta/2$, for all $k \geq K_\epsilon$.

A similar argument as above then shows that, for all $\delta > 0$, there exists some $K''_\delta \in \mathbb{N}_0$ such that

$$mk(1 - F(\theta(r/m)) - \delta/m) \leq S_{n+1} \leq mk(1 - F(\theta(r/m)) + \delta/m), \quad (25)$$

with at least probability $1 - \eta$, for all $k \geq K''_\delta$.

Now let $L_\delta := \max(K'_\delta, K''_\delta)$. We have thus proven that, for all $\delta > 0$, given $S_n = k$,

$$mk(1 - F(\theta(r/m)) - \delta/m) \leq S_{n+1} \leq mk(1 - F(\theta(r/m)) + \delta/m) \quad (26)$$

must hold with at least probability $1 - \eta$, for all $k \geq L_\delta$.

Note that, given $\Gamma_n = k \geq L_\delta$, the inequalities

$$\Gamma_{n+1} = Q^\pi(D_n(k), (X_n^k)_{k=1}^{D_n(k)}, R_n(k)) \quad (27)$$

$$\geq M(D_n(k), (X_n^k)_{k=1}^{D_n(k)}, R_n(k)) \quad (28)$$

$$\geq mk(1 - F(\theta(r/m)) - \delta/m) \quad (29)$$

hold with at least probability $1 - \eta$. Combining Equations (26) and (29) then implies that, conditioned on $\Gamma_n \geq L_\delta$ and $S_n \geq L_\delta$, the inequalities

$$\frac{\Gamma_{n+1}}{\Gamma_n} \geq m(1 - F(\theta(r/m)) - \delta/m) \geq \frac{S_{n+1}}{S_n} - 2\delta \quad (30)$$

must hold with at least probability $1 - \eta$. This proves the statement of Theorem 7.1. \square

We can now deduce the following lemma, which will be the key for Theorem 5.1.

Lemma 7.1. *Let $(\Gamma_n)_n$ be any RDBP controlled by some policy π on $(D_n^k, X_n^k, R_n^k)_{n,k}$, where this double array of i.i.d. triples of random variables satisfies assumptions (A1)-(A4) of Section 3. Then, if $q_\Gamma = 1$, we must have $q_S = 1$ too. The same result remains true in the multivariate case.*

Proof. By Theorem 5.3, we know that, given $S_n \rightarrow \infty$, $(S_n)_n$ behaves asymptotically like a supercritical GWP with some reproduction mean $\tilde{m} > 1$. Therefore, $S_{n+1}/S_n \rightarrow \tilde{m}$ a.s. Let $1 > \eta > 0$. We deduce that, for all $\delta > 0$, there exists some $L'_\delta \in \mathbb{N}_0$ such that, given $S_k \geq L'_\delta$, $S_{k+1}/S_k \geq \tilde{m} - \delta$ must hold with at least probability $1 - \eta$. Combining this with Theorem 7.1, and putting $\delta := \tilde{m}/4$, we get, given $S_n \rightarrow \infty$ and $S_k \geq \max(L_\delta, L'_\delta) =: L$,

$$\frac{\Gamma_{k+1}}{\Gamma_k} \geq \frac{S_{k+1}}{S_k} - \delta \geq \tilde{m} - 2\delta = \tilde{m}/2 \quad (31)$$

with at least probability $(1 - \eta)^2 \geq 1 - 2\eta$.

Define, for $k \in \mathbb{N}$,

$$\tau_k := \begin{cases} \inf\{N : S_n \geq k \ \forall n \geq N\}, & \text{if } \exists N \text{ such that } S_n \geq k \ \forall n \geq N, \\ +\infty, & \text{otherwise.} \end{cases} \quad (32)$$

We can thus write

$$\mathbb{P}[\Gamma_n \rightarrow \infty | \Gamma_0 = 1] \geq \mathbb{P}[S_n \rightarrow \infty | S_0 = 1] \mathbb{P}[\Gamma_n \rightarrow \infty | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1] \quad (33)$$

$$= \sum_{N=0}^{\infty} \mathbb{P}[\Gamma_n \rightarrow \infty | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1, \tau_L = N] \mathbb{P}[\tau_L = N | S_n \rightarrow \infty, S_0 = 1] \quad (34)$$

$$\geq \sum_N \sum_{M \geq N} \mathbb{P}[\Gamma_n \rightarrow \infty | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1, \tau_L = N, \Gamma_M \geq L] \quad (35)$$

$$\cdot \mathbb{P}[\Gamma_M \geq L | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1] \mathbb{P}[\tau_L = N | S_n \rightarrow \infty, S_0 = 1] \mathbb{P}[S_n \rightarrow \infty | S_0 = 1] \quad (36)$$

$$\geq (1 - 2\eta) \mathbb{P}[S_n \rightarrow \infty | S_0 = 1] \sum_N \sum_{M \geq N} \mathbb{P}[\Gamma_M \geq L | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1] \quad (37)$$

$$\cdot \mathbb{P}[\tau_L = N | S_n \rightarrow \infty, S_0 = 1]$$

Now note that $\mathbb{P}[S_n \geq L \forall n \geq M | S_n \rightarrow \infty, S_0 = 1] \nearrow 1$ as $M \rightarrow \infty$. Let $M_0 \in \mathbb{N}_0$ such that

$$\mathbb{P}[S_n \geq L \forall n \geq M_0 | S_n \rightarrow \infty, S_0 = 1] \geq 1 - \eta. \quad (38)$$

We deduce, as this sequence is increasing in M ,

$$\sum_M \mathbb{P}[\Gamma_M \geq L | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1] \mathbb{P}[S_n \geq L \forall n \geq M | S_n \rightarrow \infty, S_0 = 1] \quad (39)$$

$$\geq (1 - \eta) \sum_{M \geq M_0} \mathbb{P}[\Gamma_M \geq L | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1] \quad (40)$$

$$\geq (1 - \eta) \mathbb{P}[\exists M \geq M_0 : \Gamma_M \geq L | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1], \quad (41)$$

so that finally,

$$\mathbb{P}[\Gamma_n \rightarrow \infty | \Gamma_0 = 1] \quad (42)$$

$$\geq (1 - 3\eta) \mathbb{P}[\exists M \geq M_0 : \Gamma_M \geq L | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1] \mathbb{P}[S_n \rightarrow \infty | S_0 = 1]. \quad (43)$$

Given $S_n \rightarrow \infty$, it is clear that the process $(\Gamma_n)_n$ must be able to reach or exceed any finite level with positive probability, so that

$$\mathbb{P}[\exists M \geq M_0 : \Gamma_M \geq L | S_n \rightarrow \infty, S_0 = \Gamma_0 = 1] > 0.$$

This and Inequality (43) allow us to conclude that, if $\mathbb{P}[\Gamma_n \rightarrow \infty | \Gamma_0 = 1] = 0$, then $\mathbb{P}[S_n \rightarrow \infty | S_0 = 1] = 0$ too. The conclusion then follows from our Proposition 5.1. \square

We can now give a proof of Theorem 5.1.

Proof of Theorem 5.1. To simplify notations, write $\overline{W} := \lim_n W_n$, $\overline{\Gamma} := \lim_n \Gamma_n$ and $\overline{S} := \lim_n S_n$. As these three random variables take a.s. only the values 0 and ∞ by Proposition 5.1, we get from conditioning on $\overline{\Gamma}$,

$$\mathbb{P}[\overline{S} \leq \overline{\Gamma} \leq \overline{W}] = q_{\Gamma} \mathbb{P}[\overline{S} \leq \overline{\Gamma} \leq \overline{W} | \overline{\Gamma} = 0] + (1 - q_{\Gamma}) \mathbb{P}[\overline{S} \leq \overline{\Gamma} \leq \overline{W} | \overline{\Gamma} = \infty] \quad (44)$$

$$= q_{\Gamma} \mathbb{P}[\overline{S} = 0 | \overline{\Gamma} = 0] + (1 - q_{\Gamma}) \mathbb{P}[\overline{W} = \infty | \overline{\Gamma} = \infty] \quad (45)$$

Now, by Lemma 7.1, $q_\Gamma = 1$ implies $q_S = 1$, and thus $\bar{\Gamma} = 0$ a.s. implies $\bar{S} = 0$ a.s. Further, by Proposition 5.4, we must have $\bar{W} = \infty$ a.s. if $\bar{\Gamma} = \infty$ a.s. Consequently, the preceding probability simply becomes

$$P[\bar{S} \leq \bar{\Gamma} \leq \bar{W}] = q_\Gamma + (1 - q_\Gamma) = 1, \quad (46)$$

thus proving the result. \square

7.4 Proof of the extinction criteria

In this section, we give a proof of Theorems 5.2 and 5.3. We shall make repeatedly use of the following lemma, which we shall prove first.

Lemma 7.2. *Let X_1, X_2, \dots be i.i.d. real-valued non-negative random variables with mean $\mu < \infty$ and continuous distribution function F . Further, let $(\Phi_n)_n$ be a sequence of integer-valued random variables with $\Phi_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$, and let $(\Psi_n)_n$ be a sequence of real random variables with $\Psi_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. Suppose that $\Psi_n/\Phi_n \rightarrow \rho$ a.s. with $0 < \rho \leq \mu$, and that τ is the solution of*

$$\int_0^\tau x dF(x) = \rho.$$

Let $N(\cdot, \cdot, \cdot)$ be defined by (2). Then $N(\Phi_n, (X_k)_{k=1}^{\Phi_n}, \Psi_n)/\Phi_n \rightarrow F(\tau)$ a.s.

Proof. Since $\Psi_n/\Phi_n \rightarrow \rho > 0$ a.s. as $n \rightarrow \infty$, we have

$$\forall \epsilon > 0 : P \left[\sup_{n \geq m} |\Psi_n/\Phi_n - \rho| < \epsilon \right] \rightarrow 1 \quad \text{a.s. as } m \rightarrow \infty. \quad (47)$$

To simplify notations, we write $N(t, s) := N(t, (X_k)_{k=1}^t, s)$. As this function is stochastically increasing in s , we deduce that, for all $0 \leq \epsilon < \rho$ and all $\delta > 0$, the inequalities

$$\frac{N(\Phi_n, (\rho - \epsilon)\Phi_n)}{\Phi_n} \leq \frac{N(\Phi_n, \Psi_n)}{\Phi_n} \leq \frac{N(\Phi_n, (\rho + \epsilon)\Phi_n)}{\Phi_n} \quad (48)$$

must hold (simultaneously), for all n sufficiently large, with probability at least $1 - \delta$. Refining a result of Coffman, Flatto, and Weber [1987], Bruss and Robertson [1991] (see Theorem 2.2, p.615) have shown that

$$\frac{N(n, s_n)}{n} \rightarrow F(\tau(s)) \quad \text{a.s. as } n \rightarrow \infty \quad (49)$$

where $\tau(s)$ is the solution of

$$\int_0^{\tau(s)} x dF(x) = \lim_{n \rightarrow \infty} \frac{s_n}{n} =: s, \quad (50)$$

provided that the latter limit exists and satisfies $0 < s \leq \mu = E[X]$. Note that $\tau(\cdot)$ is of course continuous on $(0, \mu)$.

As $\Phi_n \rightarrow \infty$ a.s., the left-hand side variable of (48) must converge a.s. to $F(\tau(\rho - \epsilon))$ and the right-hand side variable a.s. to $F(\tau(\rho + \epsilon))$. Since $\epsilon > 0$ is arbitrary and $\lim_{\epsilon \rightarrow 0^+} F(\tau(\rho \pm \epsilon)) = F(\tau(\rho))$ by continuity of $F(\cdot)$ and $\tau(\cdot)$, the Lemma is proved. \square

We can now prove Theorem 5.2.

Proof of Theorem 5.2. We first prove statement (a). Suppose $r \leq m\mu$ and $mF(\tau) < 1$, and let

$$W^\infty(\Omega) = \{\omega \in \Omega : W_n(\omega) \rightarrow \infty \text{ as } n \rightarrow \infty\}. \quad (51)$$

In the following, we write $N_n(t, s) := N(t, (X_n^k)_{k=1}^t, s)$.

Now look at

$$\mathbb{E}[W_{n+1}|W_n = w] = \mathbb{E}[N_n(D_n(w), R_n(w))|W_n = w]. \quad (52)$$

Since $R_n(w)/w \rightarrow r$ a.s. as $w \rightarrow \infty$ and $D_n(w)/w \rightarrow m$ a.s. as $w \rightarrow \infty$ and $m > 0$, we have $R_n(w)/D_n(w) \rightarrow \rho = r/m$ a.s. According to Lemma 4.1 of [Bruss and Robertson \[1991\]](#), there exists a sequence $T_w \rightarrow \tau$ a.s. as $w \rightarrow \infty$ with

$$\mathbb{E}[N_n(D_n(w), R_n(w))|W_n = w] \leq \mathbb{E}[D_n(w)]F(T_w) \quad (53)$$

where τ is the solution of $\int_0^\tau x dF(x) = \rho$. Since F is continuous we can find, for each $\epsilon > 0$, a value $w = w(\epsilon)$ such that $F(T_v) \leq F(\tau + \epsilon)$ for all $v \geq w$. Thus, from equations (52) and (53):

$$\mathbb{E}[W_{n+1}|W_n = v] \leq mv F(\tau + \epsilon), \quad v \geq w \quad (54)$$

and so

$$\mathbb{E}[W_{n+1}|W_n \geq w] = \sum_{v=w}^{\infty} \mathbb{P}[W_n = v|W_n \geq w] \mathbb{E}[W_{n+1}|W_n = v] \quad (55)$$

$$\leq m F(\tau + \epsilon) \sum_{v=w}^{\infty} v \mathbb{P}[W_n = v|W_n \geq w] \quad (56)$$

$$= m F(\tau + \epsilon) \mathbb{E}[W_n|W_n \geq w]. \quad (57)$$

Since $mF(\tau) < 1$, we can choose, again by continuity of F , $\epsilon > 0$ such that $mF(\tau + \epsilon) < 1$. The latter implies then that $\mathbb{E}[W_n]$ must be bounded. Consequently, Proposition 5.1 allows us to conclude that $\mathbb{P}[W^\infty(\Omega)] = 0$, or equivalently $q_W = 1$. This proves the first part of Theorem 5.2(a).

To see the second part of Theorem 5.2(a), we now suppose that $mF(\tau) > 1$ and that $F(r/k) > 0$ for some $k \geq 2$ with $p_k > 0$. First note that under these conditions each individual can (alone) provide, with positive probability, enough resources for at least two offspring to stay. Therefore $\mathbb{P}[W_n \geq 2^{n-1}] > 0$, so that $(W_n)_n$ can reach or exceed any finite state w with positive probability. Therefore it suffices to show that

$$\exists \alpha \geq 1 : \liminf_{k \rightarrow \infty} \mathbb{P}[W_{n+k} \geq \alpha^k w | W_n \geq w] > 0. \quad (58)$$

Let $h(j, \alpha, w) := \mathbb{P}[W_{n+j} < \alpha^j w | W_{n+j-1} \geq \alpha^{j-1} w]$. Then clearly

$$\mathbb{P}[W_{n+k} \geq \alpha^k w | W_n \geq w] \geq (1 - h(k, \alpha, w)) \mathbb{P}[W_{n+k-1} \geq \alpha^{k-1} w | W_n \geq w] \quad (59)$$

and so by recurrence

$$\mathbb{P}[W_{n+k} \geq \alpha^k w | W_n \geq w] \geq \prod_{j=1}^k (1 - h(j, \alpha, w)). \quad (60)$$

Therefore a sufficient condition for (58) to hold is

$$\sum_{j=1}^{\infty} h(j, \alpha, w) < \infty \quad \text{and} \quad h(j, \alpha, w) < 1 \quad \text{for all } j. \quad (61)$$

The second condition of (61) is clearly satisfied. We now show that the first condition also holds. Since $N_n(\cdot, \cdot)$, $D_{n+j}(\cdot)$ and $R_{n+j}(\cdot)$ are stochastically increasing in their arguments, we have

$$h(j, \alpha, w) = \mathbb{P}[W_{n+j} < \alpha^j w | W_{n+j-1} \geq \alpha^{j-1} w] \quad (62)$$

$$= \mathbb{P}[N_{n+j}(D_{n+j}(W_{n+j-1}), R_{n+j}(W_{n+j-1})) < \alpha^j w | W_{n+j-1} \geq \alpha^{j-1} w] \quad (63)$$

$$\leq \mathbb{P}[N_{n+j}(D_{n+j}(\lfloor \alpha^{j-1} w \rfloor), R_{n+j}(\lfloor \alpha^{j-1} w \rfloor)) < \alpha^j w]. \quad (64)$$

Choose $\epsilon > 0$ such that $(m - \epsilon)F(\tau) > 1$ and put $\alpha = (m - \epsilon)F(\tau)$. Then, $p_0^\alpha < 1$ so that

$$\sum_{j=1}^{\infty} p_0^{\lfloor \alpha^{j-1} w \rfloor} = \sum_{j=1}^{\infty} \mathbb{P}[D_{n+j}(\lfloor \alpha^{j-1} w \rfloor) = 0] < \infty. \quad (65)$$

Since the D_{n+j} are independent random variables, it follows from the Borel-Cantelli Lemma that $\mathbb{P}[D_{n+j}(\lfloor \alpha^{j-1} w \rfloor) = 0 \text{ i.o.}] = 0$. Therefore, for j sufficiently large, inequality (64) is equivalent to

$$h(j, \alpha, w) \leq \mathbb{P}[L_j < R_j] \quad (66)$$

where

$$L_j = \frac{N_{n+j}(D_{n+j}(\lfloor \alpha^{j-1} w \rfloor), R_{n+j}(\lfloor \alpha^{j-1} w \rfloor))}{D_{n+j}(\lfloor \alpha^{j-1} w \rfloor)} \quad (67)$$

and

$$R_j = \frac{\alpha^{j-1} w}{D_{n+j}(\lfloor \alpha^{j-1} w \rfloor)}. \quad (68)$$

Put $\beta_j = \lfloor \alpha^{j-1} w \rfloor$ and look first at the random variables $D_{n+j}(\beta_j)/\beta_j \sim D(\beta_j)/\beta_j$. Recall that $D(n)$ is a sum of n i.i.d. random variables with finite mean m and finite variance σ_D^2 , say. Also, since $\alpha > 1$, $\beta_j/j \rightarrow \infty$ as $j \rightarrow \infty$. Therefore, it follows from the Hsu-Robbins theorem of complete convergence (see Theorem 1 of [Hsu and Robbins \[1947\]](#), and for furthergoing results [Asmussen and Kurtz \[1980\]](#)) that $D(\beta_j)/\beta_j \rightarrow m$ completely, and thus

$$\forall \delta > 0: \sum_{j=1}^{\infty} \mathbb{P} \left[\left| \frac{D(\beta_j)}{\beta_j} - m \right| > \delta \right] < \infty. \quad (69)$$

Further, since $\alpha^j w / \beta_j \rightarrow \alpha$ as $j \rightarrow \infty$ and $D_{n+j}(\cdot) \sim D(\cdot)$, we obtain from (68) and (69)

$$\forall \delta > 0: \sum_{j=1}^{\infty} \mathbb{P} \left[\left| R_j - \frac{\alpha}{m} \right| > \delta \right] < \infty. \quad (70)$$

Secondly, to study the convergence of L_j defined in (67) we turn to Lemma 7.2 with $\Phi_j = D(\beta_j)$ and $\Psi_j = R(\beta_j)$. Since $\Phi_j/\beta_j \rightarrow m$ completely and $\Psi_j/\beta_j \rightarrow r$ completely (again by the Hsu-Robbins theorem), we have

$$\frac{\Psi_j}{\Phi_j} \rightarrow \rho = \frac{r}{m} \quad \text{completely, as } j \rightarrow \infty. \quad (71)$$

Therefore, in particular, $\Psi_j/\Phi_j \rightarrow \rho$ a.s., so that the conditions of Lemma 7.2 are satisfied. It follows that if L_j in (67) allows for a limit (in some sense) l , say, then we must have $l = F(\tau)$, where τ is defined as in Lemma 7.2. Using this and the Chernoff-type estimates obtained by [Coffman et al. \[1987\]](#) with $a = j\delta$, we obtain after some straightforward simplifications:

$$\mathbb{P} \left[\left| \frac{N(\cdot, \cdot)}{j} - F(\tau) \right| > \delta \right] \leq 2e^{-\frac{j\delta^2}{4F(\tau)}}. \quad (72)$$

Again $N_{n+j}(\cdot, \cdot) \sim N(\cdot, \cdot)$ and $\beta_j/j \rightarrow \infty$, and thus $L_j \rightarrow F(\tau)$ completely as $j \rightarrow \infty$, implying

$$\forall \delta > 0 : \sum_{j=1}^{\infty} \mathbb{P}[|L_j - F(\tau)| > \delta] < \infty. \quad (73)$$

Now choose $\delta = \frac{1}{2}|F(\tau) - \alpha/m| > 0$. Note that the event $\{L_j < R_j\}$ cannot happen unless $|L_j - F(\tau)| > \delta$ or $|R_j - \alpha/m| > \delta$. Therefore, from (66),

$$h(j, \alpha, w) \leq \mathbb{P}[L_j < R_j] \quad (74)$$

$$\leq \mathbb{P}[|L_j - F(\tau)| > \delta] + \mathbb{P}\left[\left|R_j - \frac{\alpha}{m}\right| > \delta\right] \quad (75)$$

so that, according to (70) and (73)

$$\sum_{j=1}^{\infty} h(j, \alpha, w) < \infty. \quad (76)$$

This completes the proof of statement (a). Statement (b) is easily obtained, using Theorem 2.1 of [Bruss and Robertson \[1991\]](#). \square

Theorem 5.3 can be obtained by similar considerations, the role of τ being now played by θ , defined by $\int_{\theta}^b x dF(x) = \frac{r}{m}$, as in Theorem 2.3 of [Bruss and Robertson \[1991\]](#).

7.5 Proof of Corollaries 5.2 and 5.3

Lemma 7.3. Assume $r > \mu$ and let τ be defined by

$$\int_0^{\tau} x dF(x) = \frac{r}{m}.$$

Then, $mF(\tau) > 1$.

Proof. As $F(x) > 0$ for all $x > a$ and as $\tau > a$ (because $\int_0^{\tau} x dF(x) = \frac{r}{m} > \frac{\mu}{m} \geq 0$), we deduce $F(\tau) > 0$ and we can write

$$mF(\tau) \int_0^{\tau} x \frac{dF(x)}{F(\tau)} = r > \mu,$$

or equivalently

$$mF(\tau) \mathbb{E}[X|X \leq \tau] > \mu.$$

However, $\mathbb{E}[X|X \leq \tau] \leq \mathbb{E}[X] = \mu$, so that the above inequality cannot hold unless $mF(\tau) > 1$. \square

Using this lemma and Theorem 5.2(a)(ii), the proof of Corollary 5.2 is immediate.

Lemma 7.4. Assume $r < \mu$ and let θ be defined by

$$\int_{\theta}^{\infty} x dF(x) = \frac{r}{m}.$$

Then, $m(1 - F(\theta)) < 1$.

Proof. As $1 - F(x) > 0$ for all $x < b$ and as $\theta < b$ (because $\int_{\theta}^{\infty} x dF(x) = \frac{r}{m} > \frac{\mu}{m} \geq 0$), we deduce $1 - F(\theta) > 0$ and we can write

$$m(1 - F(\theta)) \int_{\theta}^{\infty} x \frac{dF(x)}{1 - F(\theta)} = r < \mu,$$

or equivalently

$$m(1 - F(\theta)) \mathbb{E}[X|X > \theta] < \mu.$$

However, $\mathbb{E}[X|X > \theta] \geq \mathbb{E}[X] = \mu$, so that the above inequality implies $m(1 - F(\theta)) < 1$. \square

Using this lemma and Theorem 5.3(a)(ii), the proof of Corollary 5.3 is immediate.

7.6 A remark on the unbounded case

Recall that we supposed throughout this paper that the support of F was bounded (see assumption **(A3)**). However, viewing completeness, we should at least discuss the case of unbounded support.

Certain cases are intuitive. For instance, it is clear that, with finite mean resource production and growing tails in F , the model has to allow in the limit for an infinite mean reproduction rate r . Otherwise survival would be impossible. (For infinite mean branching processes, see e.g. [Grey \[1977\]](#) and [Barbour and Schuh \[1979\]](#).) In principle, the general case of unbounded resource claims could be treated too, yet with some restriction on interpretability. The following theorem shows that the required conditions are likely to always become hard to decipher.

Theorem 7.2. *Let $(S_n)_n$ be a sf-process on $(D_n^k, X_n^k, R_n^k)_{n,k}$, where this double array of i.i.d. triples of random variables satisfies assumptions **(A1)**-**(A4)** of Section 3 but where F has now a support $[a, \infty]$, $a \geq 0$. Suppose $m > 1$ and $\mu > 0$. For all $\epsilon > 0$ and $K \in \mathbb{R}^+$, define $p_K = P(X > K)$ and*

$$\alpha_{K,\epsilon} = \inf\{F(b) - F(a) : 0 \leq a < b \leq K, b - a = \epsilon\}. \quad (77)$$

Assume that, for each $\epsilon > 0$ small enough, there exists some sequence $(K_n)_n$ such that

$$n p_{K_n} \rightarrow 0 \quad \text{and} \quad n K_n (1 - \alpha_{K_n, \epsilon})^n \rightarrow 0, \quad (78)$$

where p_K and $\alpha_{K,\epsilon}$ are defined as in (77). Then, all the results of Theorem 5.3 are true.

A new version of Theorem 2.3 of [Bruss and Robertson \[1991\]](#) is needed, in the case when the distribution is not bounded. However, Condition (78) about the tail of the distribution F is now needed.

Lemma 7.5. *Let X_1, X_2, \dots be i.i.d. continuous real-valued non-negative random variables with distribution function F and mean $\mu < \infty$. Assume F having a support $[a, \infty]$, $a \geq 0$. For all $\epsilon > 0$ and $K \in \mathbb{N}$, define p_K and $\alpha_{K,\epsilon}$ as in Theorem 7.2, and assume that, for each $\epsilon > 0$ small enough, there exists an increasing sequence $(K_n)_n$ of positive integers, such that*

$$n p_{K_n} \rightarrow 0 \quad \text{and} \quad n K_n (1 - \alpha_{K_n, \epsilon})^n \rightarrow 0. \quad (79)$$

Let $(s_n)_n$ be a sequence such that $s_n/n \rightarrow \sigma \in (0, \mu)$ as $n \rightarrow \infty$, and define θ by

$$\int_{\theta}^{\infty} x dF(x) = \sigma. \quad (80)$$

Moreover, write $M_n := M(n, s_n) := M(n, (X_k)_{k=1}^n, s_n)$ as defined by (3). Then,

$$X_{n-M_n+1,n} \xrightarrow{P} \theta \quad \text{and} \quad \frac{1}{n} M_n \xrightarrow{P} 1 - F(\theta). \quad (81)$$

Proof. Let $L = \liminf_{n \rightarrow \infty} X_{n-M_n+1,n}$. Then, using the continuity of F and the Glivenko-Cantelli theorem,

$$\int_L^{\infty} x dF(x) = \liminf_{n \rightarrow \infty} \int_{X_{n-M_n+1,n}}^{\infty} x dF(x) \quad (82)$$

$$= \liminf_{n \rightarrow \infty} \int_{X_{n-M_n+1,n}}^{\infty} x dF_n(x) \quad (83)$$

$$\leq \liminf_{n \rightarrow \infty} \frac{s_n}{n} = \sigma = \int_{\theta}^{\infty} x dF(x). \quad (84)$$

This inequality implies $\liminf_{n \rightarrow \infty} X_{n-M_n+1,n} \geq \theta$. Similarly, $\limsup_{n \rightarrow \infty} X_{n-M_n,n} \leq \theta$. In order to prove the convergence of $X_{n-M_n+1,n}$ in probability, it thus suffices to show that $X_{n-M_n+1,n} - X_{n-M_n,n}$ converges to 0 in probability.

Let $\epsilon > 0$. We write, for all K ,

$$\mathbb{P}[X_{n-M_n+1,n} - X_{n-M_n,n} > \epsilon] \quad (85)$$

$$\leq \mathbb{P}[X_{n-M_n+1,n} - X_{n-M_n,n} > \epsilon \mid X_{n-M_n+1,n} < K] + \mathbb{P}[X_{n-M_n+1,n} > K] \quad (86)$$

and we shall consider both terms separately. First,

$$\mathbb{P}[X_{n-M_n+1,n} > K] \leq \mathbb{P}[X_{n,n} > K] = 1 - (1 - p_K)^n \quad (87)$$

where $p_K := 1 - F(K)$. On the other hand, define

$$\alpha_{\epsilon,K} := \inf\{F(b) - F(a) : 0 \leq a < b \leq K, b - a = \epsilon\}. \quad (88)$$

Let $r = \lceil K/\epsilon \rceil$, and I_1, \dots, I_r be a covering of $[0, K]$ by intervals of lengths ϵ . We have, for every m ,

$$\mathbb{P}[\nexists i, j : i \neq j, X_i, X_j \in I_m] \leq (1 - \alpha_{\epsilon,K})^n + n(1 - \alpha_{\epsilon,K})^{n-1} \mathbb{P}[X \in I_m] \quad (89)$$

$$\leq (n+1)(1 - \alpha_{\epsilon,K})^{n-1}. \quad (90)$$

Therefore, we get

$$\mathbb{P}[\exists j : X_{j+1,n} - X_{j,n} > \epsilon] = \mathbb{P}[\exists m : \nexists i, j, i \neq j, X_i, X_j \in I_m] \quad (91)$$

$$\leq r(n+1)(1 - \alpha_{\epsilon,K})^{n-1} \leq \frac{K + \epsilon}{\epsilon}(n+1)(1 - \alpha_{\epsilon,K})^{n-1}. \quad (92)$$

Inequalities (86), (87) and (92) give, for all $n \in \mathbb{N}_0$ and for all $K > 0$,

$$\mathbb{P}[X_{n-M_n+1,n} - X_{n-M_n,n} > \epsilon] \leq 1 - (1 - p_K)^n + \frac{K + \epsilon}{\epsilon}(n+1)(1 - \alpha_{\epsilon,K})^{n-1}. \quad (93)$$

It thus suffices to prove that the right-hand side converges to 0 as $n \rightarrow \infty$, if K is replaced by some sequence $(K_n)_n$. Therefore, we need

$$(1 - p_{K_n})^n \rightarrow 1 \quad \text{and} \quad nK_n(1 - \alpha_{\epsilon,K_n})^n \rightarrow 0, \quad (94)$$

which is clearly satisfied if $np_{K_n} \rightarrow 0$ and $nK_n(1 - \alpha_{\epsilon,K_n})^n \rightarrow 0$, as was assumed in the statement of this result.

We have thus obtained that $X_{n-M_n+1,n} \rightarrow \theta$ in probability. Therefore, we also deduce that $M(n, s_n)/n = 1 - F_n(X_{n-M_n+1,n}) \rightarrow 1 - F(\theta)$ in probability, thanks to the Glivenko-Cantelli theorem. \square

Using this result, Lemma 7.2 is easily adapted to our case.

Lemma 7.6. *Let X_1, X_2, \dots be i.i.d. continuous real-valued non-negative random variables with distribution function F and mean $\mu < \infty$. Assume that F has a support $[a, \infty]$, $a \geq 0$, and that, for each $\epsilon > 0$ small enough, there exists some sequence $(K_n)_n$ such that*

$$np_{K_n} \rightarrow 0 \quad \text{and} \quad nK_n(1 - \alpha_{K_n,\epsilon})^n \rightarrow 0, \quad (95)$$

where p_K and $\alpha_{K,\epsilon}$ are defined as in Theorem 7.2. Further, let $(\Phi_n)_n$ be a sequence of integer-valued random variables with $\Phi_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$, and let $(\Psi_n)_n$ be a sequence of real-valued random variables with $\Psi_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. Suppose $\Psi_n/\Phi_n \rightarrow \rho$ a.s. with $0 < \rho \leq \mu$ and that θ is the solution of

$$\int_{\theta}^b x dF(x) = \rho. \quad (96)$$

Moreover, define $M(t, s) := M(t, (X_k)_{k=1}^t, s)$ as defined by (3). Then,

$$M(\Phi_n, \Psi_n)/\Phi_n \rightarrow 1 - F(\theta), \quad \text{in probability.} \quad (97)$$

The arguments of the proof of Theorem 5.2 are easily adapted to our case, now using Lemma 7.6 instead of Lemma 7.2, and yield Theorem 7.2. However, as indicated before, the interpretation of the conditions in the unbounded case seems in general difficult.

8 Conclusion

The first question posed in the Abstract is almost a philosophical one, and here Mathematics can hardly be of any help. We see the answer, as said in the Introduction (Section 1), somewhere in the middle of what contrasting theories have to offer, but this is no more than a personal perception. (Statistics could provide of course some answers in terms of a quantification of the likelihood of reasons.)

Concerning the second question our answer is implicit in our model, and hence partially subjective. We think that the *desire to survive* and to see future generations survive is the most important factor to keep any society together. The second important factor is in our view a sufficient standard of living for individuals. If individuals feel that the society could provide this but refuses to do so by sticking to a policy wasting resources by favouring too much certain groups, they are likely to emigrate. Of course, there are many other factors which may be of influence, as for instance worries that the society does not fight enough for human rights, or may not do enough for the environment, or others, but our model does not take such influences into consideration.

The third answer is complete and interesting. The wf-society and the sf-society constitute together a “global” envelope in the following sense. For given distributions of natality, resource creation and resource consumption, little can be said “locally”, as we have pointed out before. What we find very interesting is that, despite this difficulty, one can show that in the long run, there is no escape, that is, asymptotically any RDBP $(\Gamma_n)_n$ will live between the boundaries $(S_n)_n$ and $(W_n)_n$, and hence their critical parameters are of great interest. Note also that the parameters τ and θ involved in the computations of the critical values (see Subsections 5.3 and 5.4) may be seen as thresholds in the *Lorenz curve* we know from Economics.

No other society in this model does as much for ensuring survival as the wf-society. The price to pay under the same fixed distribution is the most modest standard of living of individuals in this society. The sf-society forms the other extreme. Under the given assumptions this society does the most for the standard of living of the few. However, it jeopardises the prospects of survival severely, and definitely more than any other society.

Moreover, if our perception of the weakest-first society as an extreme form of communism, and the strongest-first society as an extreme form of capitalism is seen as defensible, then we can conclude that mankind has already come close to testing the limits of societies.

Authors’ address:
Université Libre de Bruxelles
Faculté des sciences
Département de Mathématique, CP 210
B-1050 Brussels, Belgium

References

- Søren Asmussen and Thomas G. Kurtz. Necessary and sufficient conditions for complete convergence in the law of large numbers. *Ann. Probab.*, 8(1):176–182, 1980.
- Andrew D. Barbour and H.-J. Schuh. Functional normalization for the branching process with infinite mean. *J. Appl. Probab.*, 16(3):513–525, 1979.
- N. H. Bingham and R. A. Doney. Asymptotic properties of supercritical branching processes. I. The Galton-Watson process. *Advances in Appl. Probability*, 6:711–731, 1974.
- F. Thomas Bruss. Branching processes with random absorbing processes. *J. Appl. Probability*, 15(1):54–64, 1978.
- F. Thomas Bruss. A counterpart of the Borel-Cantelli lemma. *J. Appl. Probab.*, 17(4):1094–1101, 1980.
- F. Thomas Bruss and James B. Robertson. “Wald’s lemma” for sums of order statistics of i.i.d. random variables. *Adv. in Appl. Probab.*, 23(3):612–623, 1991.
- E. G. Coffman, Jr., L. Flatto, and R. R. Weber. Optimal selection of stochastic intervals under a sum constraint. *Adv. in Appl. Probab.*, 19(2):454–473, 1987.
- Harry Cohn. On the asymptotic patterns of supercritical branching processes in varying environments. *Ann. Appl. Probab.*, 6(3):896–902, 1996.
- Harry Cohn and Fima Klebaner. Geometric rate of growth in Markov chains with applications to population-size-dependent models with dependent offspring. *Stochastic Anal. Appl.*, 4(3):283–307, 1986.
- R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert W function. *Adv. Comput. Math.*, 5(4):329–359, 1996.
- M. González, M. Molina, and I. Del Puerto. On the class of controlled branching processes with random control functions. *J. Appl. Probab.*, 39(4):804–815, 2002.
- Miguel González, Manuel Molina, and Inés del Puerto. On L^2 -convergence of controlled branching processes with random control function. *Bernoulli*, 11(1):37–46, 2005.
- D. R. Grey. Almost sure convergence in Markov branching processes with infinite mean. *J. Appl. Probability*, 14(4):702–716, 1977.
- Patsy Haccou, Peter Jagers, and Vladimir A. Vatutin. *Branching processes: variation, growth, and extinction of populations*. Cambridge Studies in Adaptive Dynamics. Cambridge University Press, Cambridge, 2007.
- Sophie Hautphenne. Extinction probabilities of supercritical decomposable branching processes. *J. Appl. Prob.*, 49(3):639–651, 2012.
- P. L. Hsu and Herbert Robbins. Complete convergence and the law of large numbers. *Proc. Nat. Acad. Sci. U. S. A.*, 33:25–31, 1947.
- Peter Jagers. *Branching processes with biological applications*. Wiley-Interscience [John Wiley & Sons], London, 1975. Wiley Series in Probability and Mathematical Statistics—Applied Probability and Statistics.
- F. C. Klebaner. A limit theorem for population-size-dependent branching processes. *J. Appl. Probab.*, 22(1):48–57, 1985.

- Karl Pearson. *Tables of the incomplete beta-function*. Cambridge University Press, Cambridge, England, second edition, 1968.
- H.-J. Schuh. A condition for the extinction of a branching process with an absorbing lower barrier. *J. Math. Biol.*, 3(3-4):271–287, 1976.
- B. A. Sevast'janov and A. M. Zubkov. Controlled branching processes. *Teor. Veroyatnost. i Primen.*, 19:15–25, 1974.
- Kuang Xu and Shie Mannor. Rate-optimal control for resource-constrained branching processes. *arxiv:1203.1072v1*, 2012.
- Andrei Y. Yakovlev and Nikolai M. Yanev. Relative frequencies in multitype branching processes. *Ann. Appl. Probab.*, 19(1):1–14, 2009.
- Nikolai M. Yanev. Conditions for degeneracy of φ -branching processes with random φ . *Theory Prob. Applic.*, 20 (2):421–428, 1976.